Descartes' folii

"Curve"

$$x^3+y^3=xy$$

For each given x there are three (complex) ys which solve it. And if (x,y) solves it so does (y,x). So the "curve" is symmetric about the line $y=x$. Let's try to discover it.

Polar coordinates. Set

$$x=r\cos\theta,\quad y=r\sin\theta$$

to get

$$r^{\cancel{3}}(\cos^3\theta+\sin^3\theta)=\cancel{r^2}\cos\theta\sin\theta,$$

that is

$$r=r(\theta)=\frac{\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}.$$

This determines r as a function of θ, except where the denominator vanishes (not simultaneously with the numerator!). In many such graphical applications it's convenient to also let r have negative values.

Simplify, using

$$c := \cos\theta, \quad s := \sin\theta, \quad t := \tan\theta,$$

so

$$c^2 + s^2 = 1, \quad t = \frac{s}{c}, \quad s = ct,$$

$$c^2(1+t^2) = 1, \quad c^2 = \frac{1}{1+t^2}, \quad s^2 = \frac{t^2}{1+t^2}$$

$$c = \frac{1}{\sqrt{1+t^2}}, \quad s = \frac{t}{\sqrt{1+t^2}}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

Now

$$(c+s)^3 = c^3 + 3c^2 s + 3cs^2 + s^3,$$

that is

$$\begin{aligned} c^3 + s^3 &= (c+s)^3 - 3cs(c+s) \\ &= (c+s)\left[(c+s)^2 - 3cs\right] \\ &= (c+s)(c^2 + 2cs + s^2 - 3cs) \\ &= (c+s)(1-cs). \end{aligned}$$

Thus

$$r = \frac{cs}{(c+s)(1-cs)}$$

When does the denominator $= 0$?

By Euler

$$e^{i\theta} = c + is;$$

by trig

$$e^{i(\theta+\phi)} = e^{i\theta} e^{i\phi}.$$

that is

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi,$$
$$\sin(\theta+\phi) = \cos\theta\sin\phi + \sin\theta\cos\phi.$$

In particular,

$$\sin(2\theta) = 2cs,$$

so

$$1 - cs = 1 - \tfrac{1}{2}\sin(2\theta) \geq \tfrac{1}{2} > 0.$$

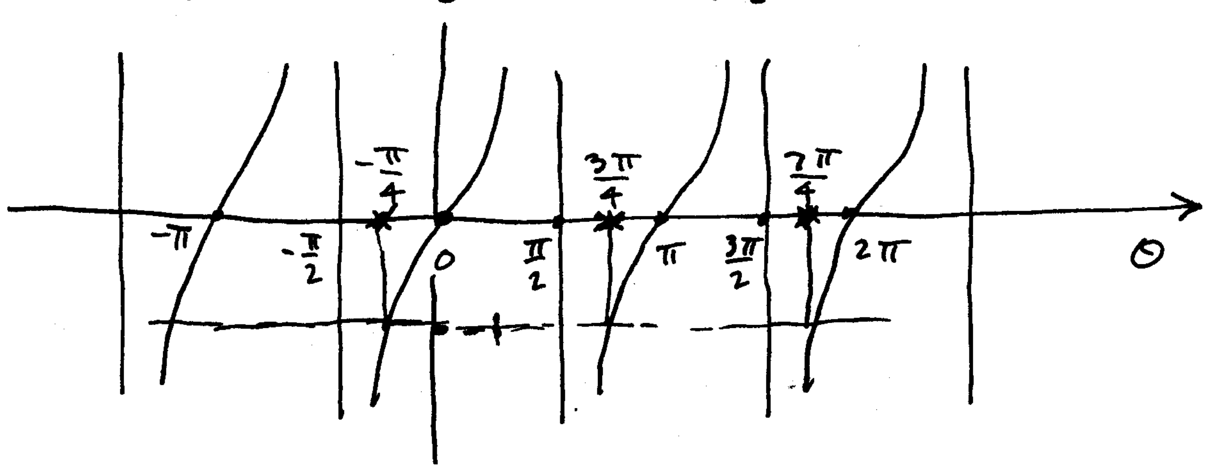
Thus the denominator $= 0$ iff (if and only if)

$$c + s = 0, \text{ or } t = \tan\theta = -1.$$

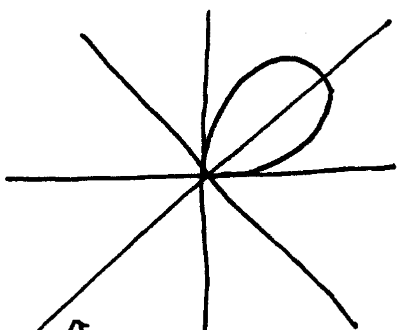
All the trig functions have period 2π, but $\tan\theta$ has the smaller period π, since

$$\tan(\theta+\pi) = \frac{\sin(\theta+\pi)}{\cos(\theta+\pi)} = \frac{\cos\theta\,\overset{0}{\cancel{\sin\pi}} + \sin\theta\,\overset{-1}{\cancel{\cos\pi}}}{\cos\theta\,\underset{-1}{\cancel{\cos\pi}} - \sin\theta\,\overset{0}{\cancel{\sin\pi}}}$$
$$= \frac{-\sin\theta}{-\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

The graph of $\tan\theta$ is



It's usual to normalize the range of θ as $0 \le \theta < 2\pi$, or as $-\pi < \theta \le \pi$. I prefer the latter, but it really doesn't matter. The values of θ where $r = r(\theta)$ "blows up" are shown on the θ-axis. Our picture of one "folium of Descartes" starts to unfold.



$$r = r(\theta) = \frac{\cos\theta \sin\theta}{(\cos\theta + \sin\theta)(1 - \cos\theta\sin\theta)}$$

$-\pi < \theta \le \pi$,

$x = x(\theta) = r(\theta)\cos\theta$,

$y = y(\theta) = r(\theta)\sin\theta$

The matlab instructions (code):

```
h=pi/2/100;  θ = 0:h:pi/2;
c=cos(θ);    s = sin(θ);
r = c.*s./(c+s)./(1-c.*s);
x = r.*c;    y = r.*s;
plot(x,y)
```

graphs a "loop" as above. This is

only part of the curve. We purposely stayed away from the θ-points where $r(\theta)$, $x(\theta)$ and $y(\theta)$ "blow up", because it's a bit fussy for matlab to deal with that.

The tangent parameterization.

All trig functions can be written in terms of $t = \tan\theta$, almost, except for the sign of the cosine; see page two.

In this problem we can write x and y in terms of t alone. This is a bit cleaner than the θ-parameterization above.

First we have

$$r = \frac{cs}{(c+s)(1-cs)} \frac{\frac{1}{c^2}}{\frac{1}{c^2}} = \frac{t}{(1+t)(\frac{1}{c}-s)}$$

$$= \frac{t}{(1+t)c\left(\frac{1}{c^2}-t\right)}, \quad c^2 = \frac{1}{1+t^2}$$

$$= \frac{1}{c}\frac{t}{(1+t)(1-t+t^2)} = \frac{1}{c}\frac{t}{1+t^3} \; !$$

DFC

So, because

$$(1+t)(1-t+t^2) = 1+t^3, \quad t = \frac{s}{c},$$

we have

$$x = \frac{t}{1+t^3}, \quad y = \frac{t^2}{1+t^3}$$

Now the <u>only</u> t-point at which x and y "blow up" is at $t=-1$. Why? This is easy to see. Look at $t = \tan\theta$ as θ moves over one period, say from 0 to 2π. Then $\tan\theta$ increases from 0 to $+\infty$, jumps instantaneously to $-\infty$ when $\theta = \frac{\pi}{2}$, and increases from $-\infty$ to 0 where $\theta = \pi$. Then it does this all over again when θ increases from π to 2π. The tan-range $(-\infty, +\infty)$ is covered twice as θ varies in $(0, 2\pi)$. Each $t = \tan\theta$ comes from <u>two</u> θs. Since we can express x and y via t this "tangent parameterization" of the curve is "more efficient". With the θ-parameterization the Descartes' folium is covered twice! It still has the same position and shape in the xy-plane. But the points, $(x,y) = (x(\theta), y(\theta))$ or $(x,y) = (x(t), y(t))$, move along the curve at different "speeds", with respect to θ, or t. The idea of parameterizing curves is very useful. It helps a lot to avoid the little technical problems related to "multivalued functions", which are not functions at all! The functions $x(\theta), y(\theta)$, or $x(t), y(t)$, above are (single valued) functions which describe the curve. If we used the (old)

$y=f(x)$ convention we would have to describe this curve via <u>three</u> (single-valued) functions f_1, f_2, f_3. And even that is a moderately intractable problem.

We can use derivatives to study parameterized curves. For instance we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if } \frac{dx}{dt} \neq 0. \qquad (*)$$

The meaning of this is easy. When $y=y(x)$ is viewed as a function of x then $y' := \frac{dy}{dx}$ has its usual meaning. But we now also let $x=x(t)$ be a function of the "timelike" variable t. Then $y=y(x(t))$ also becomes a function of t.

The chain rule, when applicable, then says that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}, \qquad (**)$$

which is (*)! There is some "shorthand" going on here which we elaborate upon. Here y is viewed both as a function of x, and a function of t. To be more clear we should write $y = y(x)$ or $y = y(t) := y(x(t))$ and carry along all the arguments.

So, (**) just says that, if $y = y(x)$ is a function of x, and $x = x(t)$ is a function of t, then

$$\frac{dy}{dt}(x(t)) = \frac{dy}{dx}(x(t))\,\frac{dx}{dt}(t),$$

that is

$$\frac{dy}{dx}(x(t)) = \frac{\frac{dy}{dt}(x(t))}{\frac{dx}{dt}(t)} \text{ if } \frac{dx}{dt}(t) \neq 0.$$

Finally, when the parameter is called t, we use

$$\overset{\circ}{x} := \frac{dx}{dt}, \quad \overset{\circ}{y} := \frac{dy}{dt},$$

to mean the <u>derivatives with respect to</u> t. Thus, the really shorthand notation for $(*)$ is

$$y' = \frac{\overset{\circ}{y}}{\overset{\circ}{x}} \quad \text{if } \overset{\circ}{x} \neq 0!$$

<u>Example</u>. Check out the slope $\frac{dy}{dx}$ for (x,y) on the folium.

$\square$ $\quad x = \dfrac{t}{1+t^3}, \quad y = \dfrac{t^2}{1+t^3}, \quad t \neq -1.$

$\Rightarrow$

$$\overset{\circ}{x} = \frac{(1+t^3)1 - t \cdot 3t^2}{(1+t^3)^2} = \frac{1-2t^3}{(1+t^3)^2},$$

$$\overset{\circ}{y} = \frac{(1+t^3)2t - t^2 \cdot 3t^2}{(1+t^3)^2} = \frac{t(2-t^3)}{(1+t^3)^2},$$

$$t \neq -1,$$

and

$$y' = \frac{\overset{\circ}{y}}{\overset{\circ}{x}} = \frac{t(2-t^3)}{1-2t^3}, \quad t \neq \frac{1}{2^{1/3}}.$$

Note well that the latter expresses

$$y' = \frac{dy}{dx}(x(t))$$ as a function of t!

Thus, for instance, we have:

Horizontal tangents to curve when $y' = 0 \iff t = 0$ or $t = 2^{1/3}$

$$\iff (x,y) = (0,0) \text{ or } (x,y) = \tfrac{1}{3}(2^{1/3}, 2^{2/3}) :=$$

$$:= \left(\frac{2^{1/3}}{3}, \frac{2^{2/3}}{3}\right), \text{ since}$$

$$x(2^{1/3}) = \frac{2^{1/3}}{1+2} = \frac{2^{1/3}}{3},$$

$$y(2^{1/3}) = 2^{1/3}\, x(2^{1/3}) = \frac{2^{2/3}}{3}.$$

Vertical tangents to curve when $y' = \pm\infty \iff t = \frac{1}{2^{1/3}} \iff$

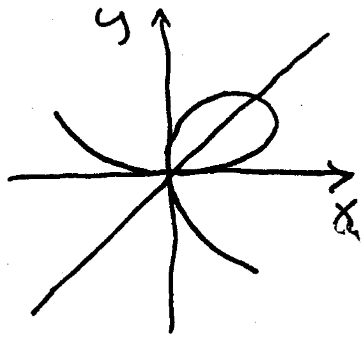
$$(x,y) = \tfrac{1}{3}(2^{2/3}, 2^{1/3}), \text{ since}$$

$$x\left(\frac{1}{2^{1/3}}\right) = \frac{\frac{1}{2^{1/3}}}{1+\frac{1}{2}} = \frac{2}{3}\,\frac{1}{2^{1/3}}$$

$$= \frac{2^{2/3}}{3} \doteq 0.5291,$$

$$y\left(\frac{1}{2^{1/3}}\right) = \frac{1}{2^{1/3}}\,\frac{2^{2/3}}{3} = \frac{2^{1/3}}{3} \doteq 0.4200.$$

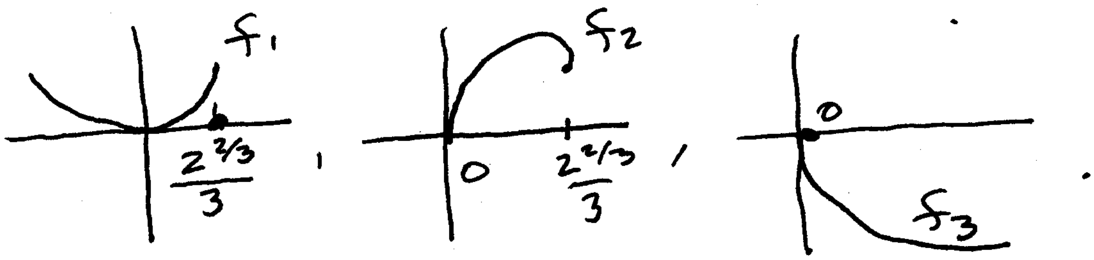
<u>What happened to the vertical tangent at $(x,y)=(0,0)$?</u> The curve actually looks as below. This is seen by using a matlab code roughly like:



line of Symmetry.

```
t1 = -5:0.01:-1.1;
t2 = -0.9:0.01:5;
t = [t1 t2];
d = 1+t.*t;
x = t./d; y = t.*x;
plot(x,y)
```

The "vertical tangent" at $(x,y)=(0,0)$ appears, on reflection(!), about the line $y=x$, as the horizontal tangent at $(x,y)=(0,0)$. The "function" $y=f(x)$ is <u>three functions</u>



The formal math breaks down at the (trifurcation) point $(x,y) = (0,0)$ where the three graphs meet. There is bifurcation at $(x,y) = \frac{1}{3}(2^{2/3}, 2^{1/3})$ where the graphs of f_1 and f_2 meet. The chain rule condition $\overset{\circ}{x} \neq 0$ must be revised:

$$y' = \frac{\overset{\circ}{y}}{\overset{\circ}{x}} \text{ if } \overset{\circ}{x} \neq 0,\ \dot{x} \neq \infty\ !$$

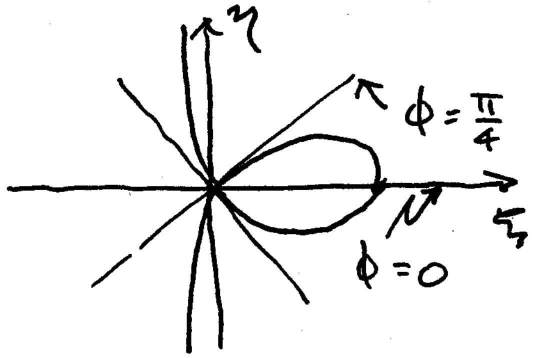
Questions re this Descartes folium.

- what are the height and width of the loop?
- what is the "asymptote", or "asymptotic line" as (x,y) goes off to (∞,∞), in the two obvious directions?
- what's the perimeter (arc length) of the curve which bounds the loop?

* what's the area of the loop?

The last two are probably <u>very hard</u>, but, <u>of course</u>, <u>they can be done numerically</u> !!!

We could (probably) solve the first two problems with the current setup, but I would rather have a double valued function instead of a three valued one! Let's first <u>rotate</u> the figure through $-\frac{\pi}{4}$ radians $=$ $= -45^\circ$ to get one looking roughly like. The (Greek) variables (ξ, η) will be the new variables. In the real world ($:=$ the world beyond Calculus),



<u>points in the plane are written as columns</u>, not rows as in Calculus. As the <u>bridge between the two notations</u> we simply

define

$$\begin{bmatrix} x \\ y \end{bmatrix} =: (x,y), \quad \begin{bmatrix} \xi \\ \eta \end{bmatrix} =: (\xi,\eta),$$

etcetera. <u>Now we can use all the power of matrix-vector notation!</u>

To <u>rotate</u> a point $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ into another point $\begin{bmatrix} \xi \\ \eta \end{bmatrix}$ in $\mathbb{R}^2$, <u>counter-clockwise</u> thru an angle θ, <u>means that</u>

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}; \quad \begin{array}{l} c = \cos\theta, \\ s = \sin\theta. \end{array}$$

This is just basic trig. We have polar coordinates, complex numbers, and two by two matrices. Why are the latter rarely mentioned? One possible answer: matrices came (much) later! The above just <u>means that</u>

$$\xi = cx - sy, \quad \eta = sx + cy.$$

<u>that's all!</u>

In our case $\theta = -\frac{\pi}{4}$ so

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} x+y \\ -x+y \end{bmatrix},$$

that is

$$\xi = \frac{\sqrt{2}}{2}(x+y), \quad \eta = \frac{\sqrt{2}}{2}(-x+y).$$

This is how we rotate (x, y) thru $\theta = -\frac{\pi}{4}$ to (ξ, η). How do we rotate back? Well,

$$x + y = \sqrt{2}\,\xi, \quad -x + y = \sqrt{2}\,\eta.$$

Adding theses gives

$$2y = \sqrt{2}(\xi + \eta);$$

subtracting gives

$$2x = \sqrt{2}(\xi - \eta).$$

Thus,

$$x = \frac{\sqrt{2}}{2}(\xi - \eta),$$

$$y = \frac{\sqrt{2}}{2}(\xi + \eta),$$

that is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{=: R(\theta),\ \theta = \frac{\pi}{4}}\begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Now

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = R(-\theta)\begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} = R(\theta)\begin{bmatrix} \xi \\ \eta \end{bmatrix},$$

and, clearly, in general,

$$\begin{aligned} R(\theta)R(-\theta) &\equiv \begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \\ &\equiv \begin{bmatrix} c^2+s^2 & cs-sc \\ sc-cs & s^2+c^2 \end{bmatrix} \\ &\equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{by Pythagoras } c^2+s^2=1. \\ &\equiv R(-\theta)R(\theta), \end{aligned}$$

check this!

So, that's how rotations work; they're <u>very</u> <u>powerful</u>!

Back to our case:

$$\xi = \frac{\sqrt{2}}{2}(x+y), \quad \eta = \frac{\sqrt{2}}{2}(-x+y).$$

We should also reparameterize the curve using the CoV (change of variable):

$$\phi := \theta - \frac{\pi}{4}, \quad \theta = \frac{\pi}{4} + \phi,$$

so that $\theta = \frac{\pi}{4}$ corresponds with $\phi = 0$, an "<u>angle</u> <u>of</u> <u>symmetry</u>":

Now put (Greek letters):

$$\gamma = \cos\phi, \quad \sigma = \sin\phi, \quad \tau = \tan\phi.$$

Then

$$\gamma = \cos\left(\theta - \frac{\pi}{4}\right)$$
$$= \cos\theta\cos\left(-\frac{\pi}{4}\right) - \sin\theta\sin\left(-\frac{\pi}{4}\right)$$
$$= \cos\theta\cos\left(\frac{\pi}{4}\right) + \sin\theta\sin\left(\frac{\pi}{4}\right)$$
$$= \frac{\sqrt{2}}{2}(c+s),$$

since cos is even, sin is odd.

Likewise,

$$\sigma = \sin\left(\theta - \frac{\pi}{4}\right)$$
$$= \cos\theta\sin\left(-\frac{\pi}{4}\right) + \sin\theta\cos\left(-\frac{\pi}{4}\right)$$
$$= -\cos\theta\sin\left(\frac{\pi}{4}\right) + \sin\theta\cos\left(\frac{\pi}{4}\right)$$
$$= \frac{\sqrt{2}}{2}(-c+s).$$

Hence,

$$\tau = \frac{\sigma}{\gamma} = \frac{-c+s}{c+s} = \frac{t-1}{t+1},$$

that is

$$\tau(t+1) = t-1, \quad 1+\tau = t(1-\tau),$$

$$t = \frac{1+\tau}{1-\tau}.$$

Substitute this into the expressions for x and y in terms of t to get x and y in terms of the new tangent parameter τ. Then <u>rotate</u> to get $\xi = \xi(\tau)$, $\eta = \eta(\tau)$, <u>as wanted</u>.

$$x = \frac{t}{1+t^3}, \quad y = \frac{t^2}{1+t^3},$$

$$1+t^3 = (1+t)(1-t+t^2),$$

$$1+t = 1+\frac{1+\tau}{1-\tau} = \frac{1-\tau+1+\tau}{1-\tau} = \frac{2}{1-\tau},$$

$$1-t = 1-\frac{1+\tau}{1-\tau} = \frac{1-\tau-1-\tau}{1-\tau} = \frac{-2\tau}{1-\tau},$$

$$\begin{aligned}
1-t+t^2 &= 1-t(1-t) \\
&= 1+\frac{1+\tau}{1-\tau}\,\frac{2\tau}{1-\tau} \\
&= \frac{(1-\tau)^2+2\tau(1+\tau)}{(1-\tau)^2} \\
&= \frac{1-\cancel{2\tau}+\tau^2+\cancel{2\tau}+2\tau^2}{(1-\tau)^2} \\
&= \frac{1+3\tau^2}{(1-\tau)^2},
\end{aligned}$$

$$1+t^3 = \frac{2}{1-\tau}\,\frac{1+3\tau^2}{(1-\tau)^2} = \frac{2(1+3\tau^2)}{(1-\tau)^3}.$$

So,

$$x = \frac{1+\tau}{\cancel{1-\tau}}\,\frac{(1-\tau)^{\cancel{3}\,2}}{2(1+3\tau^2)}$$

$$= \frac{(1+\tau)(1-\tau)(1-\tau)}{2(1+3\tau^2)}$$

$$y = t\,x = \frac{1+\tau}{1-\tau}\,x = \frac{(1+\tau)(1-\tau)(1+\tau)}{2(1+3\tau^2)},$$

$$\xi = \frac{\sqrt{2}}{2}(x+y) = \frac{\sqrt{2}}{4}\,\frac{(1+\tau)(1-\tau)}{1+3\tau^2}\,(1-\cancel{\tau}+1+\cancel{\tau})$$

$$= \frac{\sqrt{2}}{2}\,\frac{(1+\tau)(1-\tau)}{1+3\tau^2} = \frac{\sqrt{2}}{2}\,\frac{1-\tau^2}{1+3\tau^2},$$

an even function of τ, and

$$\eta = \frac{\sqrt{2}}{2}(y-x) = \frac{\sqrt{2}}{4}\,\frac{(1+\tau)(1-\tau)}{1+3\tau^2}\,(\cancel{1}+\tau-\cancel{1}+\tau)$$

$$= \frac{\sqrt{2}}{2}\,\frac{\tau(1+\tau)(1-\tau)}{1+3\tau^2} = \tau\,\xi,$$

an odd function of τ. These functions of τ, ξ and η, are defined for all τ: $-\infty < \tau < +\infty$. There are no "poles", as before. This parameterization is "wonderful" for plotting. The code

```
t = -2:0.01:2; d = 1+3*t.^2;
a = sqrt(2)/2;
x = a*(1-t.*t)./d; y = t.*x,
plot(x,y)
```

gives an elegant picture. There seem to be a vertical tangent at about $\xi \doteq 0.7$, horizontal tangents at about $\xi \doteq 0.4$, with η-values there of about ± 0.15. So the "height" of the loop is about 0.7 and the width is about 0.3. Furthermore, there is a vertical asymptote at about $\xi \doteq -0.2$, roughly. The "function" η is obviously now only a double valued "function" of ξ.

Cf., e.g., the circle $x^2+y^2=1$!

We could play games with derivatives, using now

$$\eta' = \frac{\mathring{\eta}}{\mathring{\xi}}, \quad \mathring{\xi} \neq 0 \text{ or } \infty,$$

where we cheat a little and put $\mathring{\xi} = \frac{d\xi}{d\tau}$, $\mathring{\eta} = \frac{d\eta}{d\tau}$, here. But I want to do it more directly.

I want to express η as a, double valued "function" of ξ

Now

$$\xi = a \frac{1-\tau^2}{1+3\tau^2}, \quad a := \frac{\sqrt{2}}{2},$$

$$\xi(1+3\tau^2) = a - a\tau^2,$$

$$(a+3\xi)\tau^2 = a - \xi,$$

$$\tau^2 = \tau^2(\xi) = \frac{a-\xi}{a+3\xi}.$$

We need to find the domain of this function, called $\tau^2(\xi)$, so that $\tau^2(\xi)$ is defined, and $\tau^2(\xi) \geq 0$, so $\tau := \sqrt{\tau^2}$ is real. We have

$$\frac{d}{d\xi}\tau^2(\xi) = \frac{(a+\cancel{3\xi})(-1)-(a-\cancel{\xi})3}{(a+3\xi)^2}$$

$$= \frac{-4a}{(a+3\xi)^2} < 0,$$

so $\tau^2(\xi) \searrow -\frac{1}{3}$ as $\xi \nearrow +\infty$, and $\tau^2(a) = 0$. We have $\tau^2(-\frac{a}{3}+) = +\infty$, so our function is

$$\tau^2 = \tau^2(\xi) = \frac{a-\xi}{a+3\xi}, \quad -\frac{a}{3} < \xi \le a.$$

So we may define

$$\tau = \tau(\xi) := \left(\frac{a-\xi}{a+3\xi}\right)^{1/2}, \quad -\frac{a}{3} < \xi \le a.$$

And then,

$$\boxed{\eta = \eta(\xi) = \xi\,\tau(\xi) = \xi\left(\frac{a-\xi}{a+3\xi}\right)^{1/2}, \quad -\frac{a}{3} < \xi \le a}$$

That's one "branch" of the function; the other is gotten by replacing $+\sqrt{\cdots}$ by $-\sqrt{\cdots}$.

So, clearly, the "<u>height</u>" <u>of the loop is</u> $\boxed{a = \frac{\sqrt{2}}{2} \doteq 0.7071}$. The asymptote is the line $\xi = -\frac{a}{3}$ (vertical line) in the $\xi\eta$-plane. Since $\xi = a(x+y)$ this is the line

$$\boxed{x + y = -\frac{1}{3}}$$

in the xy-plane. This line is parallel with, but not equal with, the line of "blowup" $x+y=0$!

But what's the width of the original loop. It's the height of the rotated loop, times 2. So we maximize $\eta(\xi)$ for $0 \le \xi \le a$ and multiply by 2. There is no problem with the end points since $\eta(0) = \eta(a) = 0$. So,

$$\eta' = \eta'(\xi) = \left(\frac{a-\xi}{a+3\xi}\right)^{1/2} +$$

$$+ \xi \, \frac{1}{2} \frac{\frac{(a+\cancel{3\xi})(-1)-(a-\cancel{\xi})3}{(a+3\xi)^2}}{\left(\frac{a-\xi}{a+3\xi}\right)^{1/2}}$$

$$= \left(\frac{a-\xi}{a+3\xi}\right)^{1/2} - \frac{2a\xi}{(a-\xi)^{1/2}(a+3\xi)^{3/2}}$$

$$= \left(\frac{a-\xi}{a+3\xi}\right)^{1/2}\left(1 - \frac{2a\xi}{(a-\xi)^{1/2}(a+3\xi)^{3/2}}\frac{(a+3\xi)^{1/2}}{(a-\xi)^{1/2}}\right)$$

$$= \left(\frac{a-\xi}{a+3\xi}\right)^{1/2}\left(1 - \frac{2a\xi}{(a-\xi)(a+3\xi)}\right)$$

$$= \left(\frac{a-\xi}{a+3\xi}\right)^{1/2} \frac{a^2+\cancel{2a\xi}-3\xi^2-\cancel{2a\xi}}{(a-\xi)(a+3\xi)}$$

$$= \left(\frac{a-\xi}{a+3\xi}\right)^{1/2} \frac{a^2-3\xi^2}{(a+3\xi)(a-\xi)} = 0$$

exactly when

$$\xi = \frac{a}{\sqrt{3}} = \frac{\sqrt{3}}{3}\frac{\sqrt{2}}{2} = \boxed{\frac{\sqrt{6}}{6} =: \xi^*}.$$

The maximum height occurs at

$$\xi^* = \frac{\sqrt{6}}{6} \doteq 0.4082, \text{ and is}$$

$$\eta^* = \frac{\sqrt{6}}{6}\left(\frac{\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{6}}{\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}}\right)^{1/2}$$

$$= \frac{\sqrt{6}}{6}\left(\frac{\cancel{\frac{\sqrt{2}}{2}}}{\cancel{\frac{\sqrt{2}}{2}}} \frac{1 - \frac{\sqrt{3}}{3}}{1 + \sqrt{3}} \frac{3}{3}\right)^{1/2}$$

$$= \frac{\sqrt{6}}{6}\left(\frac{1}{3} \frac{3 - \sqrt{3}}{1 + \sqrt{3}} \frac{1 - \sqrt{3}}{1 - \sqrt{3}}\right)^{1/2}$$

$$= \frac{\sqrt{2}}{6}\left(\frac{6 - 4\sqrt{3}}{1 - 3}\right)^{1/2}$$

$$= \frac{\sqrt{2}}{6}\left((2\sqrt{3} - 3)\frac{2\sqrt{3} + 3}{3 + 2\sqrt{3}}\right)^{1/2}$$

$$= \frac{\sqrt{2}}{6}\left(\frac{12 - 9}{3 + 2\sqrt{3}}\right)^{1/2}$$

$$= \frac{\sqrt{6}}{6} \frac{1}{\sqrt{3 + 2\sqrt{3}}},$$

so

$$\text{height of loop} = \frac{\sqrt{6}}{3} \frac{1}{\sqrt{3 + 2\sqrt{3}}}$$

$$\doteq 0.3211,$$

<u>as</u> <u>appears</u> <u>evident</u> <u>graphically</u>!

<u>What have we done?</u>

<u>Original equation</u>: $x^3 + y^3 = xy$

<u>CoV</u> – <u>a</u> <u>rotation</u> <u>of</u> $\mathbb{R}^2$:

$$\xi = a(x+y), \quad \eta = a(-x+y) \qquad a := \frac{\sqrt{2}}{2}$$
$$x = a(\xi - \eta), \quad y = a(\xi + \eta)$$

The original equation becomes

$$a^{\cancel{3}}(\xi-\eta)^3 + a^{\cancel{3}}(\xi+\eta)^3 = \cancel{a^2}(\xi^2-\eta^2)$$

that is

$$a\,(\xi^3 + \cancel{3\xi^2\eta} + 3\xi\eta^2 + \cancel{\eta^3}$$
$$+\xi^3 - \cancel{3\xi^2\eta} + 3\xi\eta^2 - \cancel{\eta^3}) =$$
$$= \xi^2 - \eta^2 .$$

This "kills" the η^3 term, so it becomes a quadratic in η, with coefficients depending on ξ. In fact it's the <u>trivial quadratic</u>

$$\boxed{(a+3\xi)\eta^2 = \xi^2(a-\xi)}$$

<u>That's</u> <u>all</u>! <u>Rotations</u> <u>win</u>! Now we can <u>see</u> what the <u>bi</u>furcations do! The trifurcations are gone.

Rotations again.

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} : \quad c = \cos\theta, \; s = \sin\theta .$$

Perhaps to understand this better use polar coordinates:

$$x = r\cos\phi, \; y = r\sin\phi .$$

Then

$$\begin{aligned} \xi &= r(\cos\theta\cos\phi - \sin\theta\sin\phi) \\ &= r\cos(\theta+\phi) \\ \eta &= r(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= r\sin(\theta+\phi) . \end{aligned}$$

In other words, with complex numbers,

$$\xi + i\eta = (c + is)(x + iy) .$$