<u>Problem</u>. Show that

$$\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}} = \frac{1+\sqrt{5}}{2} = g,$$

the <u>golden</u> <u>mean</u>. Let's be more careful. Let

$$a_0 := 1$$

$$a_{n+1} = \sqrt{1+a_n}, \quad n = 0,1,2,\ldots.$$

Show that

$$1 = a_0 < a_1 < \cdots < a_n \nearrow g.$$

□ First of all, if $a_n \to a^* > 0$ then, by passing to the limit in the recurrence relation, using the continuity of $\sqrt{1+a}$ for $a>0$, a^* must solve

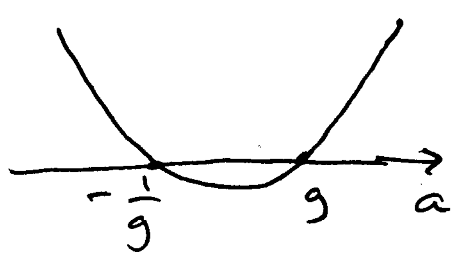
$$a = \sqrt{1+a}, \quad a > 0,$$

that is, on squaring,

$$a^2 - a - 1 = 0, \quad a > 0.$$

Thus $a = g$. But the other zero is $-\frac{1}{g}$ so we have factored

$$a^2 - a - 1 = (a-g)\left(a+\frac{1}{g}\right).$$



We have

$$a^2 - a - 1 < 0$$

for $-\frac{1}{g} < a < g$.

Now

$$a_{n+1} - a_n = \sqrt{1+a_n} - a_n$$

$$= \left(\sqrt{1+a_n} - a_n\right) \frac{\sqrt{1+a_n} + a_n}{a_n + \sqrt{1+a_n}}$$

$$= \frac{1 + a_n - a_n^2}{a_n + \sqrt{1+a_n}}$$

$$= \frac{(g - a_n)\left(\frac{1}{g} + a_n\right)}{a_n + \sqrt{1+a_n}}$$

$$> 0 \text{ if } 0 < a_n < g.$$

Let the nth error

$$e_n := g - a_n.$$

We need to show that $e_n > 0$.

Well, by a now standard trick,

$$e_{n+1} = \sqrt{1+g} - \sqrt{1+a_n}$$

$$= \frac{(\cancel{1}+g) - (\cancel{1}+a_n)}{\sqrt{1+g} + \sqrt{1+a_n}}$$

$$= \frac{e_n}{\sqrt{1+g} + \sqrt{1+a_n}}$$

with

$$e_0 = g - 1 = \frac{1}{g} > 0.$$

So all $e_n > 0$, that is all $a_n < g$.

Since $a_0 = 1 > 0$ it's a done deal. ∎

Speed of convergence.

$$\frac{e_{n+1}}{e_n} = \frac{1}{\sqrt{1+g}+\sqrt{1+a_n}} \to \frac{1}{2\sqrt{1+g}} = \frac{1}{2g}$$

$$= \frac{1}{1+\sqrt{5}}$$

$$\doteq 0.3090,$$

geometric convergence with ratio $1/(1+\sqrt{5})$.

<u>Problem</u>. Show that

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} = \frac{1+\sqrt{5}}{2}.$$

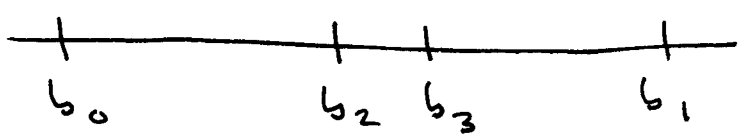
□ We have to carefully say what this means. That is, what sequence are we dealing with, <u>precisely</u>. Well, we want

$$b_0 := 1$$

$$b_1 := 1 + \frac{1}{1} = 2$$

$$b_2 := 1 + \cfrac{1}{1 + \cfrac{1}{1}} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$b_3 := 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = 1 + \frac{1}{b_2} = \frac{5}{3}.$$



What we <u>mean</u> is that

$$b_0 := 1,$$

$$b_{n+1} = 1 + \frac{1}{b_n}, \quad n = 0, 1, 2, \ldots.$$

<u>To</u> <u>prove</u>:

$$1 = b_0 < b_2 < \dots < b_{2n} \nearrow g := \frac{1+\sqrt{5}}{2},$$

the <u>golden</u> <u>mean</u>, and

$$2 = b_1 > b_3 > \dots > b_{2n+1} \searrow g .$$

We can solve for b_n <u>explicitly</u>!
We'll use some basic properties of g:

$$g^2 = 1 + g, \quad g = 1 + \frac{1}{g}, \quad g - 1 = \frac{1}{g},$$

$$g + \frac{1}{g} = 2g - 1 = \sqrt{5} .$$

CoV$_1$: let the <u>errors</u>

$$e_n :=: b_n - g, \text{ so } b_n =: g + e_n .$$

Then

$$e_{n+1} = \cancel{1} + \frac{1}{b_n} - \cancel{1} - \frac{1}{g}$$

$$= \frac{g - b_n}{g b_n} = - \frac{e_n}{g(g+e_n)} .$$

Reciprocate:

$$\frac{1}{e_{n+1}} = - \frac{1}{e_n} g(g + e_n)$$

$$= - g^2 \frac{1}{e_n} - g .$$

CoV$_2$:

$$x_n :=: \frac{1}{e_n}, \quad e_n :=: \frac{1}{x_n},$$

$$x_0 = \frac{1}{e_0} = \frac{1}{1-g} = -g .$$

Temporarily let

$$a := -g^2, \quad b = -g,$$

so as not to obscure the idea.

Then

$$x_{n+1} = a x_n + b, \quad x_0 = b .$$

By induction,

$$x_0 = b$$

$$x_1 = ab + b = (1+a)b$$

$$x_2 = a(1+a)b + b = (1+a+a^2)b$$

$$\vdots$$

$$x_n = (1 + a + a^2 + \cdots + a^n)b$$

$$= (a^{n+1} - 1)\frac{b}{a-1}$$

$$= \frac{g}{1+g^2}(a^{n+1} - 1)$$

$$= \frac{1}{g + 1/g}(a^{n+1} - 1)$$

$$= \frac{1}{\sqrt{5}}(a^{n+1} - 1) .$$

Now work backwards:

$$b_n - g = e_n = \frac{1}{x_n} = \frac{\sqrt{5}}{a^{n+1}-1}$$

$$= \sqrt{5}\ \frac{1/a^{n+1}}{1 - 1/a^{n+1}}$$

$$= \sqrt{5}\ \frac{(-1)^{n+1}\theta^{n+1}}{1+(-1)^n\theta^{n+1}}$$

$$= (-1)^{n+1}\sqrt{5}\ \frac{\theta^{n+1}}{d_n}$$

with

$$\theta := \frac{1}{g^2} = \frac{1}{1+g} = \frac{2}{3+\sqrt{5}} \doteq 0.3820$$

and positive denominators

$$d_n := 1 + (-1)^n\theta^{n+1}, \quad n = 0, 1, 2, \dots .$$

Thus

$$b_{2n} < g < b_{2n+1}.$$

Now

$$b_n - b_{n+2} = (-1)^{n+1}\sqrt{5}\,\theta^{n+1}\left(\frac{1}{d_n} - \frac{\theta^2}{d_{n+2}}\right) =$$

$$= (-1)^{n+1}\sqrt{5}\,\theta^{n+1}\ \frac{d_{n+2} - \theta^2 d_n}{d_n d_{n+2}}$$

$$= (-1)^{n+1}\sqrt{5}\,\theta^{n+1}\ \frac{1 + \cancel{(-1)^n\theta^{n+3}} - \theta^2 - \cancel{(-1)^n\theta^{n+3}}}{d_n d_{n+2}}$$

$$= (-1)^{n+1} c\ \frac{\theta^{n+1}}{d_n d_{n+2}}$$

with

$$c := \sqrt{5}(1-\theta^2) = \left(g+\frac{1}{g}\right)\left(1-\frac{1}{g^4}\right)$$

$$= \frac{1+g^2}{g}\,\frac{g^4-1}{g^4} = \frac{1+g^2}{g^5}(g-1)(1+g+g^2+g^3)$$

$$= \frac{1+g^2}{g^6}(1+g)(1+g^2) = \frac{(1+g^2)^2}{g^4}$$

$$= \frac{1}{g^2}\left(g+\frac{1}{g}\right)^2 = \frac{5}{1+g} = \frac{10}{3+\sqrt{5}} > 0 .$$

Thus,

$$b_n < b_{n+2}, \quad n \text{ even},$$

$$b_n > b_{n+2}, \quad n \text{ odd}.$$

And clearly,

$$b_n = g + (-1)^{n+1}\sqrt{5}\,\frac{\theta^{n+1}}{1+(-1)^n\theta^{n+1}}$$

$$\to g, \quad n \to +\infty,$$

since $\theta^n \to 0$, because $0 < \theta < 1$. $\blacksquare$

Every real number x has a regular continued fraction expansion (rcf),

$$x = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots}},$$

with the b_n integers and $b_n \geq 0$ for $n \geq 1$. This is gotten by a generalized Euclidean algorithm. b_0 is the integer part of x. We write $x = b_0 + \{x\}$ with x the fractional part of x: $0 \leq \{x\} < 1$. If $\{x\} = 0$ the process terminates with $x = b_0$. If $\{x\} > 0$ then one reciprocates $\{x\}$ and continues the process: $\frac{1}{\{x\}} = b_1 + \left\{\frac{1}{\{x\}}\right\}$, etcetera. Clearly, x is rational if and only if its rcf terminates.

Example.

$$\sqrt{2} \doteq 1.414$$

$$= 1 + (\sqrt{2}-1)\frac{\sqrt{2}+1}{1+\sqrt{2}} = 1 + \frac{1}{1+\sqrt{2}}$$

$$= 1 + \frac{1}{2+(\sqrt{2}-1)} = 1 + \cfrac{1}{2 + \cfrac{1}{1+\sqrt{2}}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{2+(\sqrt{2}-1)}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}},$$

so $\sqrt{2}$ is irrational: $b_0 = 1$, $b_n = 2$ for $n \geq 1$.

Problem. What about $\sqrt{m}$, $m = 3, 4, 5, \dots$? Have faith: for each m the b_n are ultimately periodic. Once they start to repeat, your job is done, for that m. Almost Schurly(!) you can't induct, on m, on this one.

Example.

$$\sqrt{3} = 1 + (\sqrt{3} - 1) = 1 + (\sqrt{3} - 1)\frac{\sqrt{3}+1}{1+\sqrt{3}}$$

$$= 1 + \frac{2}{1+\sqrt{3}} = 1 + \frac{1}{\frac{1}{2} + \frac{\sqrt{3}}{2}}$$

$$= 1 + \frac{1}{1 + \frac{\sqrt{3}-1}{2}} = 1 + \frac{1}{1 + \frac{\sqrt{3}-1}{2}\,\frac{\sqrt{3}+1}{1+\sqrt{3}}}$$

$$= 1 + \frac{1}{1 + \frac{1}{1+\sqrt{3}}} = 1 + \frac{1}{1 + \frac{1}{2 + (\sqrt{3}-1)}}$$

$$= 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{\frac{1}{2} + \frac{\sqrt{3}}{2}}}} = 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{\sqrt{3}-1}{2}}}}$$

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1+\sqrt{3}}}}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + (\sqrt{3}-1)}}}}.$$

Induction $\Rightarrow$

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cdots}}}}}.$$

That is

$b_0 = 1$

$b_1 = 1$

$b_2 = 2$

$b_3 = 1$

$\vdots$

$$\begin{cases} b_0 = 1, \\ b_{2n} = 2, & n \geq 1, \\ b_{2n+1} = 1, & n \geq 0. \end{cases}$$

a 2-periodic rcf.

Trivial example $\sqrt{4} = 2 + 0$, a terminating case: $\sqrt{4} = 2$ is rational.

Example. $\sqrt{5}$ might be interesting.

$$\sqrt{5} = 2 + (\sqrt{5}-2)\frac{\sqrt{5}+2}{2+\sqrt{5}} = 2 + \cfrac{1}{2+\sqrt{5}}$$

$$= 2 + \cfrac{1}{4+(\sqrt{5}-2)\frac{\sqrt{5}+2}{2+\sqrt{5}}} = 2 + \cfrac{1}{4+\cfrac{1}{4+(\sqrt{5}-2)}}$$

$$= 2 + \cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+\ldots}}}$$

irrational with period $p=1$.

It's rather elegant, and very classical (why don't we teach it?) that all such rcfs converge, alternatingly, as above. But the slowest convergence of all rcfs is shown by the rcf for g! (Perhaps the most basic attribute of g!)

The rcf for $e \doteq 2.718281828459046$ is "recognizable".

Problem. Find it, experimentally.

The rcf for π is "weird". So, "π is much more irrational than e"!

<u>Problem</u> (<u>Fibonacci</u> <u>numbers</u>).
Solve the <u>difference</u> <u>equation</u>

$$f_{n+1} = f_n + f_{n-1}, \quad f_0 = f_1 = 1.$$

□ This <u>recurrence</u> <u>relation</u> clearly has a unique solution. We can compute as many of the f_n as we want:

$$f_0 = f_1 = 1,$$
$$f_2 = f_1 + f_0 = 2,$$
$$f_3 = f_2 + f_1 = 3,$$
$$f_4 = f_3 + f_2 = 5,$$
$$f_5 = f_4 + f_3 = 8,$$
$$f_6 = f_5 + f_4 = 13,$$
$$f_7 = f_6 + f_5 = 21,$$
$$\vdots$$

Now they get pretty big fairly fast. Someone said this has something to do with the "breeding habits of rabbits"! The <u>sequence</u>

of integers $\{f_n\}_0^\infty$ is uniquely determined by the initial conditions $f_0 = f_1$ and the recurrence relation $f_{n+1} = f_n + f_{n-1}$, $n = 1, 2, 3, \ldots$. But what *is* f_n for an arbitrary and *general* n? How does f_n "grow" as $n \to +\infty$. Does $\lim \frac{f_{n+1}}{f_n}$ exist? What is it? What is f_{n+1}/f_n, *in general*? Well, we note that $b_n = f_{n+1}/f_n$ is the sequence of the previous problem, since $b_0 = f_1/f_0 = 1$ and $f_{n+1} = f_n + f_{n-1}$ $(n \geq 1) \iff b_n = 1 + \frac{1}{b_{n-1}}$ $(n \geq 1)$, which is the *same* recurrence relation as in that problem. So *all these problems are related*! In the previous problem we showed that

$$b_n = g + (-1)^{n+1}\sqrt{5}\,\frac{\theta^{n+1}}{d_n}$$

with positive denominators

$$d_n = 1 + (-1)^n \theta^{n+1}, \quad n = 0, 1, 2, \ldots$$

and

$$\theta = \frac{1}{g^2} = \frac{1}{1+g} = \frac{2}{3+\sqrt{5}} \doteq 0.3820.$$

We derive another formula for b_n:

$$b_n = \frac{g\left(1+(-1)^n\theta^{n+1}\right)+(-1)^{n+1}\sqrt{5}\,\theta^{n+1}}{d_n}$$

$$= \frac{g+(-1)^n\theta^{n+1}(g-\sqrt{5})}{d_n}.$$

But

$$g-\sqrt{5} = g - g - \frac{1}{g} = -\frac{1}{g},$$

so

$$b_n = \frac{g+(-1)^n\theta^{n+1}\left(-\frac{1}{g}\right)\frac{g}{g}}{d_n}$$

$$= g\,\frac{1+(-1)^{n+1}\theta^{n+2}}{d_n}$$

$$= g\,\frac{d_{n+1}}{d_n}, \quad n = 0, 1, 2, \ldots. \quad (*)$$

This is similar with

$$b_n = \frac{f_{n+1}}{f_n}, \quad n = 0, 1, 2, \ldots,$$

but it's a bit different too. From the latter we have

$$f_{n+1} = b_n f_n$$

$$f_{n+1} = b_n b_{n-1} f_{n-1}$$
$$= b_n b_{n-1} b_{n-2} f_{n-2}$$
$$= \cdots$$
$$= b_n b_{n-1} \cdots b_0 f_0$$
$$= b_0 b_1 \cdots b_n,$$

since $f_0 = 1$. That is

$$f_n = b_0 b_1 b_2 \cdots b_{n-1}, \quad n = 0, 1, 2, \ldots$$

Now use (*):

$$f_n = g \frac{\cancel{d_1}}{d_0} g \frac{\cancel{d_2}}{\cancel{d_1}} g \frac{\cancel{d_3}}{\cancel{d_2}} \cdots g \frac{d_n}{\cancel{d_{n-1}}}$$
$$= g^n \frac{d_n}{d_0}.$$

But

$$d_0 = 1 + \frac{1}{g^2} = \frac{1+g^2}{g^2} = \frac{1}{g}\left(g + \frac{1}{g}\right) = \frac{\sqrt{5}}{g},$$

so

$$f_n = \frac{\sqrt{5}}{5} g^{n+1} \left(1 + (-1)^n \theta^{n+1}\right)$$
$$= \frac{\sqrt{5}}{5} g^{n+1} \left(1 + \frac{(-1)^n}{g^{2n+2}}\right)$$

$$\boxed{f_n = \frac{\sqrt{5}}{5}\left(g^{n+1} + \frac{(-1)^n}{g^{n+1}}\right), \quad n = 0, 1, 2, \ldots}$$