<u>Halley's method</u> (1699) (<u>the</u> Halley, of comet fame).

Idea: approximate $f(x)$ locally by an <u>LFT</u> (<u>linear fractional transformation</u>):

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \cdots \\ &\doteq \frac{a+b(x-x_0)}{c+d(x-x_0)} \quad \text{near } \ x = x_0. \end{aligned}$$

Set

$$z := x - x_0, \qquad x := x_0 + z.$$

Want

$$\frac{a+bz}{c+dz} = f_0 + f_0'x + \frac{1}{2}f_0''z^2 + \cdots$$

with obious notation, that is

$$\begin{aligned} a + bz &= (c+dz)(f_0 + f_0'x + \tfrac{1}{2}f_0''z^2 + \cdots) \\ &= cf_0 + cf_0'z + \tfrac{1}{2}cf_0''z^2 + \cdots \\ &\qquad + df_0 z + df_0'z^2 + \cdots, \end{aligned}$$

so

$$a = cf_0, \quad b = cf_0' + df_0, \quad 0 = \frac{1}{2}cf_0'' + df_0'.$$

Next iterate to a zero of f is $x_1 = x_0 + \Delta x_0$ with $a + b\Delta x_0 = 0$, so $x_1 = x_0 - \frac{a}{b}$. Now

$$\frac{a}{b} = \frac{cf_0}{cf_0' + df_0} = \frac{f_0}{f_0' + \frac{d}{c}f_0}$$

so Halley's algorithm is a "correction" of Newton's. We need only d/c which follows from the coefficient of z^2 above. Since $cf_0'' + 2df_0' = 0$ then $d/c = -\frac{1}{2}f_0''/f_0'$, so the iteration is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k) - \dfrac{1}{2}\dfrac{f(x_k)f''(x_k)}{f'(x_k)}}.$$

<u>Problem</u>. Show that Halley's iteration for $f(x) = x^n - c$, $c > 0$, is <u>Lambert's algorithm of</u>

1770 for computing $\sqrt[n]{c}$:

$$x_{k+1} = x_k - \frac{(n+1)c + (n-1)x_k^n}{(n-1)c + (n+1)x_k^n}.$$

Problem. Write Halley's algorithm in the form

$$x_{k+1} = g(x_k)$$

with iteration function g. Show that $f(x^*) = 0 \Leftrightarrow x^* = g(x^*)$, so x^* is a fixed point of g. Also show that $0 = g'(x^*) = g''(x^*) \neq g'''(x^*)$, in general. Hence, with the errors $e_k = x_k - x^*$,

$$e_{k+1} = \frac{1}{6} g'''(c_k) e_k^3 \,, \qquad c_k \text{ between } x^* \text{ and } x_k,$$

so the method is of order ≥ 3.

Halley's method for inverse functions.

$$f(g(x)) \equiv x, \quad x \text{ in domain of } g$$

To compute $y = g(x)$ solve

$$F(y) := f(y) - x = 0$$

for $y = g(x)$ using Halley's algorithm.

$$y_{k+1} = y_k - \frac{F(y_k)}{F'(y_k) - \dfrac{1}{2}\dfrac{F(y_k)F''(y_k)}{F'(y_k)}}.$$

Example: $f(x) = \ln x,\ y = g(x) = e^x$.

$$y_{k+1} = y_k \frac{2 + x - \ln y_k}{2 - x + \ln y_k}, \qquad k = 0, 1, 2, \ldots.$$

Problem. What about its global convergence?
Example. $x = y_0 = 1$

$$\begin{aligned} y_0 &= 1, \\ y_1 &= 3, \\ y_2 &= 2.718064296486053, \\ y_3 &= 2.718281828459161, \\ y_4 &= 2.718281828459046, \\ y_5 &= 2.718281828459046. \end{aligned}$$