The incredible freebie, for series (theorem 3, p. 741, complex case). $c_n = a_n + i b_n$, $\sum c_n$ absolutely convergent, that is $\sum |c_n| < +\infty$, $\Rightarrow \sum c_n = \sum a_n + i \sum b_n$ convergent, that is $\sum a_n$ and $\sum b_n$ convergent.

□ Since $|c_n| := \sqrt{a_n^2 + b_n^2}$ then $|c_n| \geq |a_n|$ and $|c_n| \geq |b_n|$. We first show $\sum a_n$ convergent. Now $-|a_n| \leq a_n \leq |a_n|$, so $0 \leq a_n + |a_n| \leq 2|a_n| \leq 2|c_n|$. By assumption $\sum |c_n|$ converges. By the comparison test, so do $\sum |a_n|$ and $\sum (a_n + |a_n|)$, a series of nonnegative terms. By the limit laws, so does $\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$. Likewise, $\sum b_n$ converges ■

The root and ratio tests for absolute convergence, by comparison with geometric series.

Let $\theta \geq 0$. We know that

$$1 + \theta + \theta^2 + \theta^3 + \cdots$$

is convergent if $\theta < 1$ and divergent if $\theta > 1$, because

$$1 + \theta + \theta^2 + \cdots + \theta^{n-1} = \frac{1-\theta^n}{1-\theta}, \quad \theta \neq 1.$$

We compare the general series

$$a_1 + a_2 + a_3 + \cdots$$

with geometric series. It is essential to understand that the convergence of $\sum_1^\infty a_n$ is equivalent with the convergence of $\sum_N^{+\infty} a_n$ for some, or any, $N \geq 1$.

The root test. Look at nth roots

$$\sqrt[n]{|a_n|}, \quad n = N, N+1, \ldots.$$

There are <u>three</u> cases.

c) There are $N$, sufficiently large, and $\theta$ with

$$\sqrt[n]{|a_n|} \leq \theta < 1, \quad n \geq N$$

□ Then

$$|a_n| \leq \theta^n < 1, \quad n \geq N.$$

By the comparison test

$$\sum_N^\infty |a_n| \leq \sum_N^\infty \theta^n = \frac{\theta^N}{1-\theta} < +\infty.$$

Hence $\sum_1^\infty a_n$ is <u>absolutely convergent</u>, and so is <u>convergent.</u> ■

d) There are $N$, sufficiently large, and $\theta$ with

$$\sqrt[n]{|a_n|} \geq \theta > 1, \quad n \geq N.$$

□ Then

$$|a_n| \geq \theta^n > 1, \quad n \geq N,$$

$\theta^n \to +\infty$, so $a_n \not\to 0$ and $\sum a_n$ <u>diverges</u>. ■

i) The inconclusive case when neither of the above hypotheses can be satisfied.

<u>The ratio test</u>. Look at the ratios

$$\frac{|a_{n+1}|}{|a_n|}, \quad n = N, N+1, \cdots,$$

assuming of course that N is so large that $a_n \neq 0$ for $n \geq N$!

There are <u>three</u> cases.

c) There are N, sufficiently large, and $\Theta$ with

$$\frac{|a_{n+1}|}{|a_n|} \leq \Theta < 1, \quad n \geq N$$

□ Then

$$|a_{n+1}| \leq \Theta |a_n|, \quad n \geq N.$$

By induction,

$$|a_{N+1}| \leq \Theta |a_N|$$

$$|a_{N+2}| \leq \Theta |a_{N+1}| \leq \Theta^2 |a_N|$$

$$\vdots$$

$$|a_{N+n}| \leq \Theta^n |a_N|, \quad n = 0, 1, 2, \ldots,$$

<u>that is</u>

$$|a_n| \le \theta^{n-N}|a_N|, \quad n \geq N.$$

Thus

$$\sum_N^\infty |a_n| \le |a_N| \sum_N^\infty \theta^{n-N}$$

$$= |a_N| \sum_0^\infty \theta^n = \frac{|a_N|}{1-\theta} < +\infty,$$

$\sum_1^\infty a_n$ is <u>absolutely convergent</u>, and hence <u>convergent</u>. ■

d) There are $N$, sufficiently large, and $\theta$ with

$$\frac{|a_{n+1}|}{|a_n|} \geq \theta > 1, \quad n \geq N.$$

□ Then

$$|a_{n+1}| \geq \theta |a_n|, \quad n \geq N,$$

so (by induction)

$$|a_{N+n}| \geq \underbrace{\theta^n \underbrace{|a_N|}_{\neq 0!}}_{\to +\infty}, \quad n \geq 0.$$

Hence $|a_n| \to +\infty$ so $\sum_1^\infty a_n$ <u>diverges</u>. ■

i) The inconclusive case when neither of the above hypotheses hold.

Weak corollaries.

Root test. Suppose

$$\theta := \lim \sqrt[n]{|a_n|}$$

exists. Then:

c) $\theta < 1 \Rightarrow \sum a_n$ absolutely convergent.

d) $\theta > 1 \Rightarrow \sum a_n$ divergent

i) $\theta = 1$ is inconclusive.

Ratio test. Suppose $a_n \neq 0$ for all large $n$ and

$$\theta := \lim \frac{|a_{n+1}|}{|a_n|}$$

exists. Then:

c) $\theta < 1 \Rightarrow \sum a_n$ absolutely convergent.

d) $\theta > 1 \Rightarrow \sum a_n$ divergent

i) $\theta = 1$ is inconclusive

Very important remark. All these results apply to complex series $\sum a_n$, $a_n = \mathrm{Re}\, a_n + i\, \mathrm{Im}\, a_n$!!!

Examples – power series – can always take $z = 1$ to get ordinary series.

$$f(z) = \sum_0^\infty c_n z^n, \quad c_n, z \text{ in } \mathbb{C}, \text{ or } \mathbb{R}!$$

We want to eventually justify the "$f(z) =$" part of this. For now just use $f(z)$ as a name for the series, for shortness.

① $c_n = 1 + (-1)^n$, $a_n = c_n z^n$

Strictly speaking, the ratio test cannot be applied, since $a_{2n+1} \equiv 0$, all $n$. But the root test can: $\sqrt[n]{|a_n|} = \sqrt[n]{c_n}\, |z|$ and

$$c_n = 2, \; n \text{ even}$$
$$\equiv 0, \; n \text{ odd}$$

so

$$\sqrt[n]{c_n} = 2^{1/n} \to 1, \quad n \text{ even},$$
$$\equiv 0 \to 0, \quad n \text{ odd}.$$

Here $\sqrt[n]{c_n}$ has two limit points, 1 and 0. The largest limit point of a sequence is called its lim sup, or $\overline{\lim}$ (it could be $+\infty$). Here $\overline{\lim}\sqrt[n]{c_n} = 1$. The series converges absolutely when

$$|z| \, \overline{\lim} \sqrt[n]{|c_n|} < 1,$$

that is when

$$|z| < R := \frac{1}{\overline{\lim} \sqrt[n]{|c_n|}}$$

In the present case $R = 1$ so

$$f(z) = \sum_0^\infty (1 + (-1)^n) z^n, \quad |z| < 1.$$

Of course we already knew this, via geometric series. In fact, $f(z) = \frac{1}{1-z} + \frac{1}{1+z} = \frac{2z}{1-z^2}$ for $|z|<1$. We could use the ratio test on $f(z) = 2\sum_0^\infty z^{2n}$, but the root test always works!