Chapter 18

Mathematical Induction

EVERYONE IS FAMILIAR with the process of reasoning, called *ordinary* or *incomplete induction*, in which a generalization is made on the basis of a number of simple observations.

EXAMPLE 1. We observe that $1 = 1^2$, $1 + 3 = 4 = 2^2$, $1 + 3 + 5 = 9 = 3^2$, $1 + 3 + 5 + 7 = 16 = 4^2$, and conclude that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

or, in words, the sum of the first n odd integers is n^2.

EXAMPLE 2. We observe that 2 points determine $1 = \frac{1}{2} \cdot 2(2 - 1)$ line; that 3 points, not on a line, determine $3 = \frac{1}{2} \cdot 3(3 - 1)$ lines; that 4 points, no 3 on a line, determine $6 = \frac{1}{2} \cdot 4(4 - 1)$ lines; that 5 points, no 3 on a line, determine $10 = \frac{1}{2} \cdot 5(5 - 1)$ lines; and conclude that n points, no 3 on a line, determine $\frac{1}{2}n(n - 1)$ lines.

EXAMPLE 3. We observe that for $n = 1, 2, 3, 4, 5$ the values of

$$f(n) = \frac{n^4}{8} - \frac{17n^3}{12} + \frac{47n^2}{8} - \frac{103n}{12} + 6$$

are 2, 3, 5, 7, 11, respectively, and conclude that $f(n)$ is a prime number for every positive integral value of n.

The conclusions in Examples 1 and 2 are valid as we shall prove later. The conclusion in Example 3 is false since $f(6) = 22$ is not a prime number.

MATHEMATICAL INDUCTION or complete induction is a type of reasoning by which such conclusions as were drawn in the above examples may be proved or disproved.

The steps are

(*1*) The verification of the proposed formula or theorem for some positive integral value of n, usually the smallest. (Of course, we would not attempt to prove an unknown theorem by mathematical induction without first verifying it for several values of n.)

(*2*) The proof that if the proposed formula or theorem is true for $n = k$, some positive integer, it is true also for $n = k + 1$.

(*3*) The conclusion that the proposed formula or theorem is true for all values of n greater than the one for which verification was made in Step *1*.

EXAMPLE 4. Prove: $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

(*1*) The formula is true for $n = 1$ since $1 = 1^2$.

(*2*) Let us assume the formula true for $n = k$, any positive integer; that is, let us assume that

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2 \tag{18.1}$$

We wish to show that, when (*18.1*) is true, the proposed formula is then true for $n = k + 1$; that is, that

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2 \tag{18.2}$$

[NOTE: Statements (*18.1*) and (*18.2*) are obtained by replacing n in the proposed formula by k and $k + 1$, respectively. Now it is clear that the left member of (*18.2*) can be obtained from the left member of (*18.1*) by adding $(2k + 1)$. At this point the proposed formula is true or false according as we do or do not obtain the right member of (*18.2*) when $(2k + 1)$ is added to the right member of (*18.1*).]

Adding $(2k+1)$ to both members of (*18.1*), we have

$$1+3+5+\cdots+(2k-1)+(2k+1)=k^2+(2k+1)=(k+1)^2 \tag{18.3}$$

Now (*18.3*) is identical with (*18.2*); thus, if the proposed formula is true for any positive integer $n=k$ it is true for the next positive integer $n=k+1$.

(*3*) Since the formula is true for $n=k=1$ (Step 1), it is true for $n=k+1=2$; being true for $n=k=2$, it is true for $n=k+1=3$; and so on. Hence, the formula is true for all positive integral values of n.

Solved Problems

Prove by mathematical induction.

18.1 $1+7+13+\cdots+(6n-5)=n(3n-2)$.

(*1*) The proposed formula is true for $n=1$, since $1=1(3-2)$.

(*2*) Assume the formula to be true for $n=k$, a positive integer; that is, assume

$$1+7+13+\cdots+(6k-5)=k(3k-2) \tag{1}$$

Under this assumption we wish to show that

$$1+7+13+\cdots+(6k-5)+(6k+1)=(k+1)(3k+1) \tag{2}$$

When $(6k+1)$ is added to both members of (*1*), we have on the right

$$k(3k-2)+(6k+1)=3k^2+4k+1=(k+1)(3k+1)$$

hence, if the formula is true for $n=k$ it is true for $n=k+1$.

(*3*) Since the formula is true for $n=k=1$ (Step 1), it is true for $n=k+1=2$; being true for $n=k=2$ it is true for $n=k+1=3$; and so on, for every positive integral value of n.

18.2 $1+5+5^2+\cdots+5^{n-1}=\frac{1}{4}(5^n-1)$.

(*1*) The proposed formula is true for $n=1$, since $1=\frac{1}{4}(5-1)$.

(*2*) Assume the formula to be true for $n=k$, a positive integer; that is, assume

$$1+5+5^2+\cdots+5^{k-1}=\tfrac{1}{4}(5^k-1) \tag{1}$$

Under this assumption we wish to show that

$$1+5+5^2+\cdots+5^{k-1}+5^k=\tfrac{1}{4}(5^{k+1}-1) \tag{2}$$

When 5^k is added to both members of (*1*), we have on the right

$$\tfrac{1}{4}(5^k-1)+5^k=\tfrac{5}{4}(5^k)-\tfrac{1}{4}=\tfrac{1}{4}(5\cdot 5^k-1)=\tfrac{1}{4}(5^{k+1}-1)$$

hence, if the formula is true for $n=k$ it is true for $n=k+1$.

(*3*) Since the formula is true for $n=k=1$ (Step 1), it is true for $n=k+1=2$; being true for $n=k=2$ it is true for $n=k+1=3$; and so on, for every positive integral value of n.

18.3 $\dfrac{5}{1\cdot 2\cdot 3}+\dfrac{6}{2\cdot 3\cdot 4}+\dfrac{7}{3\cdot 4\cdot 5}+\cdots+\dfrac{n+4}{n(n+1)(n+2)}=\dfrac{n(3n+7)}{2(n+1)(n+2)}$.

(*1*) The formula is true for $n=1$, since $\dfrac{5}{1\cdot 2\cdot 3}=\dfrac{1(3+7)}{2\cdot 2\cdot 3}=\dfrac{5}{6}$.

(2) Assume the formula to be true for $n = k$, a positive integer; that is, assume

$$\frac{5}{1\cdot 2\cdot 3}+\frac{6}{2\cdot 3\cdot 4}+\cdots+\frac{k+4}{k(k+1)(k+2)}=\frac{k(3k+7)}{2(k+1)(k+2)} \qquad (1)$$

Under this assumption we wish to show that

$$\frac{5}{1\cdot 2\cdot 3}+\frac{6}{2\cdot 3\cdot 4}+\cdots+\frac{k+4}{k(k+1)(k+2)}+\frac{k+5}{(k+1)(k+2)(k+3)}=\frac{(k+1)(3k+10)}{2(k+2)(k+3)} \qquad (2)$$

When $\dfrac{k+5}{(k+1)(k+2)(k+3)}$ is added to both members of (1), we have on the right

$$\begin{aligned}\frac{k(3k+7)}{2(k+1)(k+2)}+\frac{k+5}{(k+1)(k+2)(k+3)} &= \frac{1}{(k+1)(k+2)}\left[\frac{k(3k+7)}{2}+\frac{k+5}{k+3}\right]\\ &= \frac{1}{(k+1)(k+2)}\,\frac{k(3k+7)(k+3)+2(k+5)}{2(k+3)}\\ &= \frac{1}{(k+1)(k+2)}\,\frac{3k^3+16k^2+23k+10}{2(k+3)}\\ &= \frac{1}{(k+1)(k+2)}\,\frac{(k+1)^2(3k+10)}{2(k+3)}\\ &= \frac{(k+1)(3k+10)}{2(k+2)(k+3)};\end{aligned}$$

hence, if the formula is true for $n = k$ it is true for $n = k + 1$.

(3) Since the formula is true for $n = k = 1$ (Step 1), it is true for $n = k + 1 = 2$; being true for $n = k = 2$, it is true for $n = k + 1 = 3$; and so on, for all positive integral values of n.

18.4 $x^{2n} - y^{2n}$ is divisible by $x + y$.

(1) The theorem is true for $n = 1$, since $x^2 - y^2 = (x - y)(x + y)$ is divisible by $x + y$.

(2) Let us assume the theorem true for $n = k$, a positive integer; that is, let us assume

$$x^{2k} - y^{2k} \text{ is divisible by } x + y. \qquad (1)$$

We wish to show that, when (1) is true.

$$x^{2k+2} - y^{2k+2} \text{ is divisible by } x + y. \qquad (2)$$

Now $x^{2k+2} - y^{2k+2} = (x^{2k+2} - x^2y^{2k}) + (x^2y^{2k} - y^{2k+2}) - x^2(x^{2k} - y^{2k}) + y^{2k}(x^2 - y^2)$. In the first term $(x^{2k} - y^{2k})$ is divisible by $(x + y)$ by assumption, and in the second term $(x^2 - y^2)$ is divisible by $(x + y)$ by Step 1; hence, if the theorem is true for $n = k$, a positive integer, it is true for the next one $n = k + 1$.

(3) Since the theorem is true for $n = k = 1$, it is true for $n = k + 1 = 2$; being true for $n = k = 2$, it is true for $n = k + 1 = 3$; and so on, for every positive integral value of n.

18.5 The number of straight lines determined by $n > 1$ points, no three on the same straight line, is $\frac{1}{2}n(n - 1)$.

(1) The theorem is true when $n = 2$, since $\frac{1}{2}\cdot 2(2 - 1) = 1$ and two points determine one line.

(2) Let us assume that k points, no three on the same straight line, determine $\frac{1}{2}k(k - 1)$ lines.

When an additional point is added (not on any of the lines already determined) and is joined to each of the original k points, k new lines are determined. Thus, altogether we have $\frac{1}{2}k(k - 1) + k = \frac{1}{2}k(k - 1 + 2) = \frac{1}{2}k(k + 1)$ lines and this agrees with the theorem when $n = k + 1$.

Hence, if the theorem is true for $n = k$, a positive integer greater than 1, it is true for the next one $n = k + 1$.

(3) Since the theorem is true for $n = k = 2$ (Step 1), it is true for $n = k + 1 = 3$; being true for $n = k = 3$, it is true for $n = k + 1 = 4$; and so on, for every possible integral value >1 of n.

Supplementary Problems

Prove by mathematical induction, n being a positive integer.

18.6 $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$

18.7 $1 + 4 + 7 + \cdots + (3n - 2) = \frac{1}{2}n(3n - 1)$

18.8 $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{1}{3}n(4n^2 - 1)$

18.9 $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$

18.10 $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2$

18.11 $1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{1}{30}n(n + 1)(2n + 1)(3n^2 + 3n - 1)$

18.12 $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$

18.13 $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n - 1)(2n + 1)} = \dfrac{n}{2n + 1}$

18.14 $1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + n \cdot 3^n = \frac{3}{4}[(2n - 1)3^n + 1]$

18.15 $\dfrac{3}{1 \cdot 2 \cdot 2} + \dfrac{4}{2 \cdot 3 \cdot 2^2} + \dfrac{5}{3 \cdot 4 \cdot 2^3} + \cdots + \dfrac{n + 2}{n(n + 1)2^n} = 1 - \dfrac{1}{(n + 1)2^n}$

18.16 A convex polygon of n sides has $\frac{1}{2}n(n - 3)$ diagonals.

18.17 The sum of the interior angles of a regular polygon of n sides is $(n - 2)180°$.