9 Oct 03 RT1

Rectangle (midpoint) and trapezoidal rules for $\ln x = \int_1^x \frac{dt}{t}$, $x > 1$. $\boxed{\text{DRAFT}}$

More general. Rectangle and trapezoidal rules for

$$If = \int_a^b f(t)\,dt, \quad -\infty < a < b < +\infty.$$

Step:

$$h = \frac{b-a}{n}, \quad n = (\text{large}) \text{ integer} > 0.$$

Mesh points:

$$x_k = a + kh, \quad k = 0, 1, \ldots, n.$$

Thus

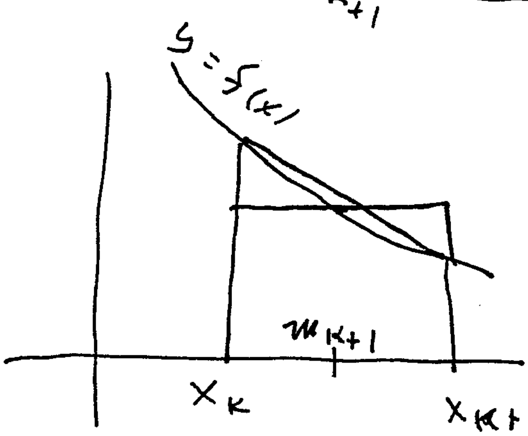
$$a = x_0 < x_1 < \cdots < x_n = b,$$

with mesh spacing

$$\Delta x_k := x_{k+1} - x_k \equiv h.$$

Midpoints:

$$m_{k+1} := \frac{x_k + x_{k+1}}{2} = x_k + \frac{h}{2}$$



Midpoint rule:

area $\doteq h f(m_{k+1})$.

Trapezoidal rule:

area $\doteq \frac{h}{2}\left[f(x_k) + f(x_{k+1})\right]$.

$$Mf(h) = h\left[f(m_1) + f(m_2) + \cdots + f(m_n)\right]$$

$$Tf(h) = h\left[\tfrac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \tfrac{1}{2}f(x_n)\right]$$

<u>Special case</u>: $f(t) = \frac{1}{t}$, $a = 1$, $b = x > 1$.

$$Mf(x,h) = h\left(\frac{1}{1+\frac{h}{2}} + \frac{1}{1+\frac{3h}{2}} + \cdots + \frac{1}{x-\frac{h}{2}}\right),$$

$$Tf(x,h) = h\left(\frac{1}{2} + \frac{1}{1+h} + \frac{1}{1+2h} + \cdots + \frac{1}{x-h} + \frac{1}{2x}\right),$$

with

$$h = \frac{x-1}{n}, \quad n = \text{integer} > 0.$$

<u>Theorem</u>. For $f(t) = \frac{1}{t}$, $x > 1$ and $h > 0$:

① $$h \cdot \frac{1}{m_{k+1}} < \int_{x_k}^{x_{k+1}} \frac{dt}{t} < \frac{h}{2}\left(\frac{1}{x_k} + \frac{1}{x_{k+1}}\right)$$

② $$Mf(x,h) < \int_1^x \frac{dt}{t} < Tf(x,h)$$

③ $$0 < Tf(x,h) - Mf(x,h) \to 0, \quad h \to 0$$

□ ① says that

$$\frac{h}{x+\frac{h}{2}} < \ln(x+h) - \ln x < \frac{h}{2}\left(\frac{1}{x} + \frac{1}{x+h}\right)$$

(for $x = x_k$). Now

$$\ln(x+h) = \ln\left[x\left(1+\frac{h}{x}\right)\right]$$
$$= \ln x + \ln\left(1+\frac{h}{x}\right)$$

so ① says

$$\frac{h}{x\left(1+\frac{h}{2x}\right)} < \ln\left(1+\frac{h}{x}\right) < \frac{h}{2x}\left(1+\frac{1}{1+\frac{h}{x}}\right).$$

That is, replacing ~~h by h/x~~ h/x by h, ①
says

$$\boxed{\frac{2h}{2+h} < \ln(1+h) < \frac{h}{2}\,\frac{2+h}{1+h}, \quad h>0}$$

Earlier we showed the weaker result that

$$\frac{1}{1+h} < \frac{\ln(1+h)}{h} < 1, \quad h > 0.$$

A matlab plot shows the stronger result to be true! Can we prove it? First, differentiate the three terms involved

$$\frac{d}{dh}\,\frac{2h}{2+h} = \frac{(2+\cancel{h})2 - \cancel{2h\cdot 1}}{(2+h)^2} = \frac{4}{(2+h)^2}$$
$$= \frac{1}{\left(1+\frac{h}{2}\right)^2}$$

$$\frac{d}{dh}\ln(1+h) = \frac{1}{1+h}$$

$$\frac{d}{dh}\,\frac{h}{2}\,\frac{2+h}{1+h} = \frac{1}{2}\frac{d}{dh}\,h\left(1+\frac{1}{1+h}\right)$$

$$= \frac{1}{2}\left(1+\frac{(1+\cancel{h})1-\cancel{h\cdot 1}}{(1+h)^2}\right)$$

$$= \frac{1}{2}\left(1+\frac{1}{(1+h)^2}\right)$$

Let

$$\phi(h) := \frac{h}{2}\,\frac{2+h}{1+h} - \ln(1+h), \quad h \geq 0.$$

Then

$$\phi(0) = 0$$

and

$$\phi'(h) = \frac{1}{2}\left(1+\frac{1}{(1+h)^2}\right) - \frac{1}{1+h}$$

$$= \frac{1}{2} + \frac{1-2(1+h)}{(1+h)^2}$$

$$= \frac{1}{2} - \frac{1+2h}{(1+h)^2}$$

$$= \frac{(1+h)^2 - 1 - 2h}{2(1+h)^2}$$

$$= \frac{\cancel{1}+\cancel{2h}+h^2-\cancel{1}-\cancel{2h}}{2(1+h)^2}$$

$$\phi'(h) = \frac{h^2}{2(1+h)^2} > 0, \quad h > 0.$$

Thus, $\phi(0) = 0$ and ϕ is <u>strictly increasing</u> for $h > 0$, so $\phi(h) > 0$ for $h > 0$, that is

$$\boxed{\ln(1+h) < \frac{h}{2}\,\frac{2+h}{1+h}, \quad h > 0}.$$

Let

$$\psi(h) := \ln(1+h) - \frac{2h}{2+h}, \quad h \geq 0.$$

Then

$$\psi(0) = 0$$

and

$$\psi'(h) = \frac{1}{1+h} - \frac{1}{\left(1+\frac{h}{2}\right)^2}$$

$$= \frac{\left(1+\frac{h}{2}\right)^2 - (1+h)}{(1+h)\left(1+\frac{h}{2}\right)^2}$$

$$= \frac{\cancel{1} + \cancel{h} + \frac{h^2}{4} - \cancel{1} - \cancel{h}}{(1+h)\left(1+\frac{h}{2}\right)^2}$$

$$= \frac{h^2}{(1+h)(2+h)^2} > 0, \quad h > 0.$$

So, $\varphi(0)=0$ and $\varphi(h)$ strictly increases for $h>0$, and so $\varphi(h)>0$ for $h>0$, that is

$$\boxed{\frac{2h}{2+h} < \ln(1+h), \quad h>0}.$$

① is proved and ② follows immediately - set $h \leftarrow \frac{h}{x_k}$ and sum over k.

For ③ we look at the differences

$$0 < \frac{h}{2}\frac{2+h}{1+h} - \frac{2h}{2+h} = h\left(\frac{1}{2}\frac{2+h}{1+h} - \frac{2}{2+h}\right)$$

$$= h\,\frac{(2+h)^2-4(1+h)}{2(1+h)(2+h)}$$

$$= h\,\frac{\cancel{4}+\cancel{4h}+h^2-\cancel{4}-\cancel{4h}}{2(1+h)(2+h)}$$

$$= \frac{h^3}{2(1+h)(2+h)}, \quad h>0$$

$$< \frac{h^3}{4}, \quad h>0$$

(because $1+h \geq 1$, $2+h \geq 2$!) Replace h by $\frac{h}{x}$:

$$0 < \frac{h}{2}\left(\frac{1}{x} + \frac{1}{x+h}\right) - h\,\frac{2}{x+(x+h)} < \frac{h^3}{4x^3}\ ,\ h>0$$

$$\leq \frac{h^3}{4}\ ,\ x \geq 1$$

Now set $x = x_k$ and sum: for $f(t) = 1/t$ and $a = 1$, $b = x \geq 1$

$$0 < Tf(x,h) - Mf(x,h) =$$

$$= h \sum_{k=0}^{n-1} \left[\frac{1}{2}\left(\frac{1}{x_k} + \frac{1}{x_{k+1}}\right) - \frac{2}{x_k + x_{k+1}}\right]$$

$$< \frac{h^3}{4} \sum_{k=0}^{n-1} \frac{1}{x_k^3} \leq \frac{h^3}{4} \sum_{k=0}^{n-1} 1 =$$

$$< \frac{nh^3}{4} = \frac{nh \cdot h^2}{4} = \frac{(x-1)h^2}{4}$$

$$\longrightarrow 0,\ h \to 0$$

$\blacksquare$

<u>**Proof**</u> that t-error := <u>local</u> T-error > <u>local</u> M-error =: m-error <u>for</u> $f(t) = \frac{1}{t}$, i.e. for $\ln x = \int_1^x \frac{dt}{t}$, $x > 1$.

□ $$\frac{h}{2}\,\frac{2+h}{1+h} - \ln(1+h) \overset{?}{>} \ln(1+h) - \frac{2h}{2+h}$$

(we know both these are >0).

$\iff$

$$\frac{h}{2}\,\frac{2+h}{1+h} + \frac{2h}{2+h} \overset{?}{>} 2\ln(1+h)$$

$\iff$

$$\boxed{\ln(1+h) \overset{?}{<} \frac{1}{2}\underbrace{\left(\frac{h}{2}\,\frac{2+h}{1+h} + \frac{2h}{2+h}\right)}, \quad h>0}$$

= average of midpoint + trapezoidal (local) approximations!

Let

$$\chi(h) := \frac{1}{2}\left(\frac{h}{2}\,\frac{2+h}{1+h} + \frac{2h}{2+h}\right) - \ln(1+h)$$

to prove $\chi(h) > 0$ for $h > 0$.

Clearly, $\chi(0) = 0$, and the result will be true if $\chi'(h) > 0$ for $h > 0$. By earlier work

$$x'(h) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2(1+h)^2} + \frac{4}{(2+h)^2}\right) - \frac{1}{1+h}$$

$$= \frac{1}{2}\,\frac{(1+h)^2(2+h)^2 + (2+h)^2 + 8(1+h)^2}{2(1+h)^2(2+h)^2} - \frac{1}{1+h}$$

$$= \frac{(1+h)^2(2+h)^2 + (2+h)^2 + 8(1+h)^2 - 4(1+h)(2+h)^2}{4(1+h)^2(2+h)^2}$$

Gather terms showing factor of $(2+h)^2$:

$$(2+h)^2\big((1+h)^2 + 1 - 4 - 4h\big) =$$

$$= (2+h)^2(1 + 2h + h^2 - 3 - 4h)$$

$$= (2+h)^2(-2 - 2h + h^2)$$

$$= -(4 + 4h + h^2)(2 + 2h - h^2)$$

$$\begin{array}{r}
2 + 2h - h^2 \\
4 + 4h + h^2 \\
\hline
8 + 8h - 4h^2 \qquad\qquad\quad \\
+8h + 8h^2 - 4h^3 \qquad \\
+2h^2 + 8h^3 - h^4 \\
\hline
8 + 16h + 6h^2 - 2h^3 - h^4
\end{array}$$

So,

$$x'(h) = \frac{\cancel{-8}\cancel{-16h} - 6h^2 + 2h^3 + h^4 + \cancel{8} + \cancel{16h} + 8h^2}{4(1+h)^2(2+h)^3}$$

$$= \frac{2h^2 + 2h^3 + h^4}{4(1+h)^2(2+h)^2} > 0, \; h > 0,$$

as desired. ∎

Is

$$\frac{m\text{-error}}{t\text{-error}} = \frac{\ln(1+h) - \frac{2h}{2+h}}{\frac{h}{2}\frac{2+h}{1+h} - \ln(1+h)}$$

$$< M \doteq \frac{1}{2}, \quad 0 < h \le h_0 .$$

As $h \to 0$ it assumes the indeterminate form $\frac{0}{0}$. Try to apply "<u>L'Hospital's</u> <u>rule</u>".

$$\frac{d}{dh}\left(\ln(1+h) - \frac{2h}{2+h}\right) = \frac{1}{1+h} - \frac{(2+\cancel{h})2 - 2\cancel{h}\cdot 1}{(2+h)^2}$$

$$= \frac{1}{1+h} - \frac{4}{(2+h)^2}$$

$$= \frac{(2+h)^2 - 4(1+h)}{(1+h)(2+h)^2}$$

$$= \frac{\cancel{4} + \cancel{4h} + h^2 - \cancel{4} - \cancel{4h}}{(1+h)(2+h)^2}$$

$$= \frac{h^2}{(1+h)(2+h)^2}$$

$$\frac{d}{dh}\left(\frac{h}{2}\frac{2+h}{1+h} - \ln(1+h)\right) = \frac{1}{2}\frac{d}{dh}h\left(1 + \frac{1}{1+h}\right) - \frac{1}{1+h}$$

$$= \frac{1}{2}\left(1 + \frac{(1+\cancel{h})1 - \cancel{h}\cdot 1}{(1+h)^2}\right) - \frac{1}{1+h}$$

$$= \frac{1}{2} + \frac{1}{2(1+h)^2} - \frac{1}{1+h}$$

$$= \frac{(1+h)^2 + 1 - 2(1+h)}{2(1+h)^2}$$

$$= \frac{\cancel{2} + \cancel{2h} + h^2 - \cancel{2} - \cancel{2h}}{2(1+h)^2}$$

$$= \frac{h^2}{2(1+h)^2},$$

so

$$\frac{\frac{d}{dh}\left(\ln(1+h) - \frac{2h}{2+h}\right)}{\frac{d}{dh}\left(\frac{h}{2}\,\frac{2+h}{1+h} - \ln(1+h)\right)} = \frac{\cancel{h^2}}{\cancel{(1+h)}(2+h)^2}\;\frac{2(1+h)^{\cancel{2}}}{\cancel{h^2}}$$

$$= \frac{2(1+h)}{(2+h)^2}$$

$$\to \frac{2}{4} = \frac{1}{2}, \quad h \to 0.$$

A matlab plot of $\frac{\text{m-error}}{\text{t-error}}(h)$ now shows this ratio to be (positive and) strictly decreasing for $h > 0$, with limit $\frac{1}{2}$ as $h \searrow 0$. Hence, experimentally,

$$0 < \frac{\text{m-error}}{\text{t-error}} < \frac{1}{2}, \quad h > 0,$$

a rather slick result. But it awaits a complete proof! "L-Hospital" follows from Taylor series!

We have

$$\ln(1+h) = h - \frac{h^2}{2} + \frac{h^3}{3} - \cdots, \quad |h| < 1$$

$$\frac{2h}{2+h} = \frac{h}{1+\frac{h}{2}} = h\left(1 - \frac{h}{2} + \left(\frac{h}{2}\right)^2 - \cdots\right)$$

$$= h - \frac{h^2}{2} + \frac{h^3}{4} - \cdots, \quad |h| < 2$$

$$\frac{h}{2}\,\frac{2+h}{1+h} = \frac{h}{2}\left(1 + \frac{1}{1+h}\right)$$

$$= \frac{h}{2}\left(1 + 1 - h + h^2 - h^3 + \cdots\right)$$

$$= \frac{h}{2}\left(2 - h + h^2 - h^3 + \cdots\right)$$

$$= h - \frac{h^2}{2} + \frac{h^3}{2} - \frac{h^4}{2} + \cdots, \quad |h| < 1$$

Hence,

$$\ln(1+h) - \frac{2h}{2+h} = \cancel{h} - \cancel{\frac{h^2}{2}} + \frac{h^3}{3} - \cdots$$

$$- \cancel{h} + \cancel{\frac{h^2}{2}} - \frac{h^3}{4} + \cdots$$

$$= \left(\frac{1}{3} - \frac{1}{4}\right) h^3 + \cdots$$

$$= \frac{1}{12} h^3 + \cdots, \quad |h| < 1$$

$$\frac{h}{2}\,\frac{2+h}{1+h} - \ln(1+h) = \cancel{h} - \cancel{\frac{h^2}{2}} + \frac{h^3}{2} - \cdots$$

$$- \cancel{h} + \cancel{\frac{h^2}{2}} - \frac{h^3}{3} + \cdots$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) h^3 + \cdots$$

$$= \frac{1}{6} h^3 + \cdots, \quad |h| < 1$$

$$\frac{\text{m-error}}{\text{t-error}} = \frac{\ln(1+h) - \frac{2h}{2+h}}{\frac{h}{2}\frac{2+h}{1+h} - \ln(1+h)}$$

$$= \frac{\frac{1}{12}h^3 + \cdots}{\frac{1}{6}h^3 + \cdots}, \quad |h| < 1$$

$$= \frac{\frac{1}{12} + O(h)}{\frac{1}{6} + O(h)}, \quad h \to 0$$

$$\to \frac{1}{2}, \quad h \to 0.$$

We don't need the <u>infinite</u> series, just the first "several" terms, with derivative form of the remainder, generalizing <u>Rolle's theorem</u> and the <u>mean value theorem</u>, which are rather clear geometrically. See text.

last

~~One more~~ time! We claim that

$$t\text{-error} := \frac{h}{2}\frac{2+h}{1+h} - \ln(1+h) >$$

$$> 2\left(\ln(1+h) - \frac{2h}{2+h}\right) =: 2(m\text{-error})$$

for $h>0$. Now let

$$\chi(h) := \frac{h}{2}\frac{2+h}{1+h} - \ln(1+h) -$$

$$- 2\left(\ln(1+h) - \frac{2h}{2+h}\right)$$

$$= \frac{h}{2}\frac{2+h}{1+h} + \frac{4h}{2+h} - 3\ln(1+h), \; h \geq 0$$

Clearly $\chi(0) = 0$. Now

$$\chi'(h) = \frac{1}{2}\frac{d}{dh}h\left(1+\frac{1}{1+h}\right) +$$

$$+ 4\frac{d}{dh}\frac{h}{2+h} - 3\frac{d}{dh}\ln(1+h)$$

$$= \frac{1}{2}\left(1 + \frac{(1+\cancel{h})\cdot 1 - \cancel{h\cdot 1}}{(1+h)^2}\right) +$$

$$+ 4\,\frac{(2+\cancel{h})\cdot 1 - \cancel{h\cdot 1}}{(2+h)^2} - \frac{3}{1+h}$$

$$= \frac{1}{2} + \frac{1}{2(1+h)^2} + \frac{8}{(2+h)^2} - \frac{6(1+h)}{2(1+h)^2}$$

$$= \frac{(1+h)^2 + 1 - 6(1+h)}{2(1+h)^2} + \frac{8}{(2+h)^2}$$

$$= \frac{2+2h+h^2-6-6h}{2(1+h)^2} + \frac{8}{(2+h)^2}$$

$$= \frac{8}{(2+h)^2} - \frac{4+4h-h^2}{2(1+h)^2}$$

$$= \frac{16(1+h)^2 - (4+4h-h^2)(4+4h+h^2)}{2(1+h)^2(2+h)^2}$$

$$= \frac{\cancel{16(1+h)^2} - \cancel{(4+4h)^2} + h^4}{2(1+h)^2(2+h)^2}$$

$$= \frac{h^4}{2(1+h)^2(2+h)^2} > 0, \quad h > 0,$$

$x(h)$ increases from 0 as h increases from 0, so $x(h) > 0$ for $h > 0$. So, <u>in fact</u>,

$$\boxed{\ln(1+h) < \frac{1}{3}\left(\frac{h}{2}\,\frac{2+h}{1+h} + \frac{4h}{2+h}\right), \quad h > 0}$$

<u>Just checking</u> the derivatives at $h = 0$:

$$x(h) = \frac{h}{2}\left(1 + \frac{1}{1+h}\right) + \frac{2h}{1+\frac{h}{2}} - 3\ln(1+h)$$

$$= \frac{h}{2}\left(1 + 1 - h + h^2 - h^3 + \cdots\right)$$

$$+ 2h\left(1 - \frac{h}{2} + \frac{h^2}{4} - \frac{h^3}{8} + \cdots\right)$$

$$- 3\left(h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \cdots\right)$$

$$= \cancel{h} - \cancel{\frac{h^2}{2}} + \cancel{\frac{h^3}{2}} - \cancel{\frac{h^4}{2}} + \frac{h^5}{2} - \cdots$$

$$+ \cancel{2h} - \cancel{h^2} + \cancel{\frac{h^3}{2}} - \cancel{\frac{h^4}{4}} + \frac{h^5}{8} - \cdots$$

$$- \cancel{3h} + \cancel{\frac{3}{2}h^2} - \cancel{h^3} + \cancel{\frac{3}{4}h^4} - \frac{3}{5}h^5 + \cdots$$

$$= \left(\frac{5}{8} - \frac{3}{5}\right)h^5 + \cdots = \frac{1}{40}h^5 + \cdots, \quad |h| < 1.$$

This means that the right side of (*) is an <u>incredibly good approximation to</u> $\ln(1+h)$ <u>for small</u> h. We have

$$\frac{1}{3}\left(\frac{h}{2}\,\frac{2+h}{1+h}+\frac{4h}{2+h}\right)-\ln(1+h)=\frac{1}{120}h^5+\cdots,$$

as $h \to 0$.

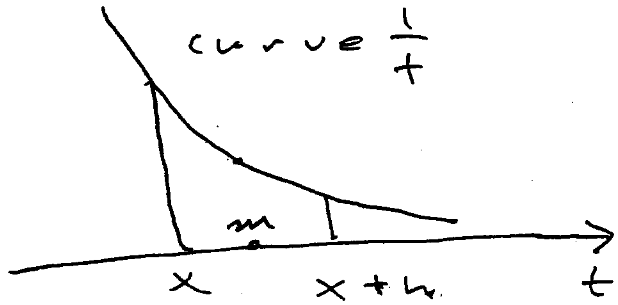
For $h = 0.1$ we have $\frac{1}{120}h^5 = \frac{0.00001}{120} =$
$\overset{x}{=} 0.000000083333\cdots = 8.\overline{3}\cdot 10^{-8}$.

This was a byproduct of a rather <u>careful</u> study of $\ln x$!

It would be nice to have a lower bound for $\ln(1+h)$ which is similarly accurate. The <u>new questions never stop</u>! If they are <u>elegant</u> questions their answers will live forever. This is what the university, research <u>and</u> teaching, is all about!

And there's more about $\ln$, <u>of course</u>!

Local stuff again.



$$m(x,h) = h\,\frac{1}{x+\frac{h}{2}}$$

$$t(x,h) = \frac{h}{2}\left(\frac{1}{x}+\frac{1}{x+h}\right)$$

Reduction of # of variables:

$$\underbrace{m(x,h)}_{\text{fn of 2 vars}} = \frac{h}{x}\,\frac{1}{1+\frac{h}{2x}} = \underbrace{m\left(\frac{h}{x}\right)}_{\text{fn of 1 var}}$$

$$t(x,h) = \frac{h}{2x}\left(1+\frac{1}{1+\frac{h}{x}}\right) = t\left(\frac{h}{x}\right)$$

$$m(h) = \frac{h}{1+\frac{h}{2}} = \frac{2h}{2+h}$$

$$t(h) = \frac{h}{2}\left(1+\frac{1}{1+h}\right) = \frac{h}{2}\,\frac{2+h}{1+h}$$

Checking that $m(h) < t(h)$, $h>0$:

$$t(h)-m(h) = h\left(\frac{2+h}{2(1+h)} - \frac{2}{2+h}\right)$$

$$= h\,\frac{(2+h)^2-4(1+h)}{2(1+h)(2+h)}$$

$$= h\,\frac{\cancel{4}+\cancel{4h}+h^2-\cancel{4}-\cancel{4h}}{2(1+h)(2+h)}$$

$$= \frac{h^3}{2(1+h)(2+h)} > 0,\quad h>0.$$

(How do we prove that $M(h) \nearrow$ and $T(h) \searrow$ as $h \searrow 0$? A little subtle!)

Replace h by $\frac{h}{x_k}$ and sum from $k=0$ to $k=n-1$

$$0 < \sum_{k=0}^{n-1}\left[t\left(\frac{h}{x_k}\right) - m\left(\frac{h}{x_k}\right)\right]$$

$$= Tf(x,h) - Mf(x,h)$$

$$= \sum_{k=0}^{n-1} \frac{h^3}{2x_k^3\left(1+\frac{h}{x_k}\right)\left(2+\frac{h}{x_k}\right)}$$

$$\leq \frac{h^3}{4}\sum_{k=0}^{n-1}\frac{1}{x_k^3} \leq \frac{h^3}{4}\cdot n =$$

$$\leq \frac{nh\cdot h^2}{4} = \frac{(x-1)h^2}{4} \to 0, \; h \to 0$$

But my matlab code dmromberg gives

h	M	T
1	0.750000	0.666667
1/2	0.708333	0.695714
1/4	0.697023	0.691220
1/8	0.694122	0.692661
1/16	0.693391	0.693025
1/32	0.693208	0.693117
1/64	0.693162	0.693140

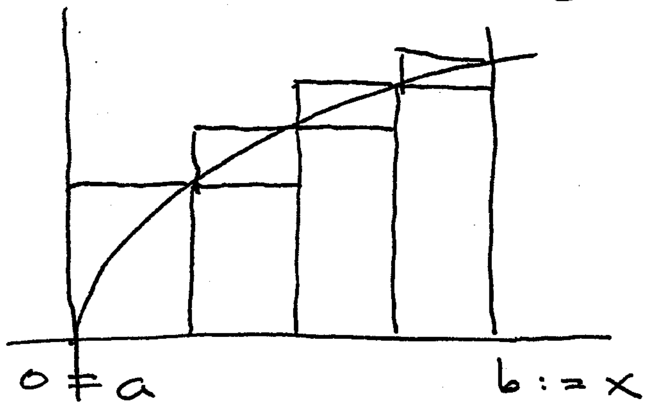
What's wrong? Answer: the output is <u>mislabeled</u>! Why?

~~Codes in r://common/greggs/work~~

Codes in r://common/greggs/work. May need to "map a drive" to get (to) them.

Example $f(t) = t^p, \ t \geq 0, \ p \geq 0$

$:= e^{p \ln t}$



$f'(t) = p t^{p-1}, \ t > 0$

$$h = \frac{b-a}{n} = \frac{x}{n}$$

$$E_n := E(h) = h \sum_0^{n-1} f(x_k)$$

$$= h \sum_0^{n-1} f(kh)$$

$$= h \sum_0^{n-1} (kh)^p$$

$$= h^{p+1} \underbrace{\sum_0^{n-1} k^p}$$

$$= n \text{ if } p = 0$$

$$= \sum_1^{n-1} k^p, \ p > 0$$

$$F_n := F(h) = h \sum_1^n f(kh)$$

$$= h^{p+1} \sum_1^n k^p$$

$$E(h) < \int_0^x t^p dt < F(h)$$

$$\int_0^x t^p\,dt = \frac{t^{p+1}}{p+1}\Big|_0^x$$

$$= \frac{x^{p+1}}{p+1}. \qquad (*)$$

$$= h^{p+1}\,\frac{n^{p+1}}{p+1}.$$

$\therefore$ (therefore)

$$\sum_0^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_1^{n} k^p \qquad (**)$$

Can prove this by induction, on n. Then $(*)$ follows.

But, if we can prove $(*)$ another way, then $(**)$ follows. What's the way?

Which came first, the chicken or the egg?