Newton's method for inverse functions.

$$f(g(x)) \equiv x \,, \quad x \text{ in domain of } g$$

Can compute $y = g(x)$ by solving

$$F(y) := f(y) - x = 0$$

for y, provided we can compute $f(y)$ and $f'(y)$, and start close enough to $g(x)$. Now

$$F'(y) = f'(y)$$

so the Newton iteration is

$$y_{k+1} = y_k - \frac{F(y_k)}{F'(y_k)}$$

$$= y_k - \frac{f(y_k) - x}{f'(y_k)}$$

Main example. I can compute $f(x) := \ln x$ by an extension of Archimedes' method for computing π! And $f'(x) = \frac{1}{x}$ here. So we compute $y = e^x$ by

iterating

$$y_{k+1} = y_k - \frac{\ln y_k - x}{1/y_k}$$

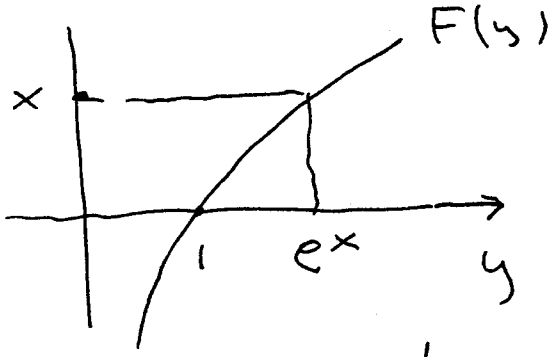
$$= y_k - y_k(\ln y_k - x)$$

$$= y_k(1 + x - \ln y_k).$$

Since $e^{-x} = \frac{1}{e^x}$ we can restrict $x > 0$. Then $e^x > 1$ so $y_0 := 1$ is a <u>lower bound</u> for e^x. We're solving

$$F(y) = \ln y - x = 0$$

for y. Look at $F(y)$! We have



$$F'(y) = \frac{1}{y},$$

$$F''(y) = -\frac{1}{y^2}$$

$F(y)$ is increasing and concave for $y > 0$.

This seems, graphically at least, to indicate that

$$0 < 1 = y_0 < y_1 < \dots < y_k \nearrow e^x, \quad k \to +\infty.$$

<u>Prove it</u>! Show $y_k < e^x \Rightarrow y_{k+1} < e^x$.