Partial Solution Set, Leon §3.2

3.2.1 Determine whether the following are subspaces of $\mathbf{R}^2$.

(b) $S = \{(x_1, x_2)^T | x_1 x_2 = 0\}$ No, this is not a subspace. Every element of S has at least one component equal to 0. The set is closed under scalar multiplication, but not under addition. For example, both $(1,0)^T$ and $(0,1)^T$ are elements of S, but their sum is not.

(c) $S = \{(x_1, x_2)^T | x_1 = 3x_2\}$. Yes, this is a subspace. To see this, let $\mathbf{x} = (3x, x)^T$ and $\mathbf{y} = (3u, u)^T$. Both are elements of S. For any scalar α, $\alpha\mathbf{x} = (3\alpha x, \alpha x) \in S$. Also $\mathbf{x} + \mathbf{u} = (3x + 3u, x + u)^T = (3(x + u), x + u)^T \in S$.

3.2.3 Determine whether the following are subspaces of $R^{2\times 2}$.

(a) The set of all 2×2 diagonal matrices is a subspace of $R^{2\times 2}$, since a scalar multiple of a diagonal matrix is diagonal and the sum of two diagonal matrices is diagonal.

(c) The set of all 2×2 matrices such that $a_{12} = 1$ is not a subspace of $R^{2\times 2}$, since it is closed under neither scalar multiplication nor addition.

3.2.4 Determine the nullspace of each of the following matrices.

(b) $A = \begin{bmatrix} 1 & 2 & -3 & -1 \\ -2 & -4 & 6 & 3 \end{bmatrix}$.

Solution: We must solve the homogeneous equation, $A\mathbf{x} = \mathbf{0}$. After elimination, we have an equivalent system, $B\mathbf{x} = \mathbf{0}$, where $B = \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

If $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$, we have $x_4 = 0$ and x_2, x_3 free. Setting $x_2 = s$ and $x_3 = t$ and applying backsubstitution, we have $x_1 = 3t - 2s$. So $\mathbf{x} = (3t - 2s, s, t, 0)^T$. It will prove useful in future work to write $\mathbf{x}$ as a linear combination of two vectors, i.e.,

$$\mathbf{x} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

3.2.5 Determine whether the following are subspaces of P_4.

(c) The set of all polynomials $p(x) \in P_4$ such that $p(0) = 0$.

Solution: If $p(0) = 0$, then $p(x) = ax^3 + bx^2 + cx$, i.e., $p(x)$ has a zero constant term. Clearly any scalar multiple inherits this property, as does the sum of of any two such polynomials. Thus this set is a subspace of P_4.

(d) The set of all polynomials in P_4 having at least one real root.

Solution: No, this is not a subspace, since closure under addition fails. For example, both $p(x) = x^2 - 2x + 1$ and $q(x) = 2x$ have real roots, but $(p+q)(x) = x^2 + 1$ has no real root.

3.2.6 Determine which of the following are subspaces of $C[-1, 1]$.

(b) The set of odd functions in $C[-1, 1]$.

Solution: Yes, this is a subspace. Let f be an odd function in $C[-1, 1]$, let $\alpha \in \mathbf{R}$, and let $x \in [-1, 1]$. Then

$$(\alpha f)(-x) = \alpha(f(-x)) = \alpha(-f(x)) = -\alpha f(x) = -(\alpha f)(x),$$

showing closure under scalar multiplication. We also have closure under addition: if g is another odd function in $C[-1, 1]$, then

$$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f+g)(x),$$

so $f + g$ is odd.

(c) The set of nondecreasing functions in $C[-1, 1]$.

Solution: No, this set is not closed under scalar multiplication. For example, $f(x) = x$ is in this set, but $(-1)f(x) = -x$ is a strictly decreasing function. Closure fails.

3.2.9 Determine whether the following are spanning sets for $\mathbf{R}^2$.

(a) $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$

Solution: Yes, this set spans $\mathbf{R}^2$. To see this, let $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbf{R}^2$. We must find (or at least show that we can find) constants α_1, α_2 such that $\mathbf{x} = \alpha_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. This equation leads to a linear system

$$\begin{array}{rcrcl} 2\alpha_1 & + & 3\alpha_2 & = & a \\ \alpha_1 & + & 2\alpha_2 & = & b \end{array}.$$

The corresponding coefficient matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ is nonsingular, so we know that a solution exists. Applying Gaussian elimination, we find $\mathbf{x} = \begin{bmatrix} 2a - 3b \\ 2b - a \end{bmatrix}$.

(b) $\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right\}$. This is not a spanning set. Setting up the linear system as in (a), we find that $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in \text{Span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right)$ if and only if $b = \dfrac{3a}{2}$.

3.2.11 Given

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} -9 \\ -2 \\ 5 \end{bmatrix},$$

(a) Is $\mathbf{x} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$?
Solution: No. The corresponding matrix equation is inconsistent.

3.2.13 In $\mathbf{R}^{2\times 2}$, let

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Show that $\{E_{ii} | 1 \leq i \leq 4\}$ spans $\mathbf{R}^{2\times 2}$.

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{R}^{2\times 2}$. Then $A = aE_{11} + bE_{12} + cE_{21} + dE_{22}$. □

3.2.14 Which of the following are spanning sets for P_3? Justify your answers.

(b) $\{2, x^2, x, 2x + 3\}$ spans P_3, since $ax^2 + bx + c = a(x^2) + b(x) + \frac{c}{2}(2)$.

(d) $\{x + 2, x^2 - 1\}$ does not span P_3. For example, $x^2 \notin \text{Span}(x + 2, x^2 - 1)$. This is easily verified using Gaussian elimination.

MA/Ra, 9/29/04