Partial Solution Set, Leon §3.3

3.3.1 Determine whether the following vectors are linearly independent in $\mathbf{R}^2$.

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Yes. These are clearly not scalar multiples of one another, and when testing two vectors that's all that we need to show.

(c) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. No. This can be shown in two ways. First, the easy way: If the span of the first two vectors is all of $\mathbf{R}^2$ (it is; they are linearly independent), all three cannot help being linearly dependent. Done.

The almost-as-easy way: Set up a homogeneous system in which the three vectors in question are the columns of a matrix A. Then apply Gaussian elimination to show that there are nontrivial solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. (Recommended, if you feel that you need more practice.)

3.3.2 Same as (1), except that we are now in $\mathbf{R}^3$. A pair of vectors is linearly independent unless they are scalar multiples of one another, and that takes care of (e). In (b), even if we can find three vectors that are linearly independent (and we can), it is easy to show that those three span $\mathbf{R}^3$, so if we add any vector(s) we create a linearly dependent set. In other words, a set of four vectors from $\mathbf{R}^3$ is inevitably linearly dependent. This leaves (a), (c), and (d). In each, let the vectors in question be the columns of a matrix A, and investigate the existence of nontrivial solutions to $A\mathbf{x} = \mathbf{0}$. The answers will be no in both (c) and (d), but yes in (a).

3.3.3 This is straightforward. A set containing only various multiples of a single nonzero vector in $\mathbf{R}^3$ (part (d), for example) generates a line through the origin. A set containing exactly two linearly independent vectors in $\mathbf{R}^3$ (e.g., parts (c) and (e)) generates a plane through the origin, and a linearly independent set of three vectors in $\mathbf{R}^3$ generates all of $\mathbf{R}^3$.

3.3.4 Now we're in $\mathbf{R}^{2\times 2}$, and things are less obvious at first glance. (They were obvious before, right?) Determine whether the following vectors are linearly independent in $\mathbf{R}^{2\times 2}$:

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. We must decide whether there is a nontrivial linear combination of these that produces the matrix of zeros. So consider a sum of the form

$$a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & a \end{bmatrix}.$$

For this to be identical to the zero matrix, we need $a = b = c = 0$, so only the trivial linear combination suffices. These vectors (yes, they're matrices, but in $\mathbf{R}^{2\times 2}$ these are *vectors*) are linearly independent in $\mathbf{R}^{2\times 2}$.

3.3.6 We are to determine whether a collection of vectors (polynomials, in this case) is linearly independent in P_3.

(a) $1, x^2$, and $x^2 - 2$ are linearly dependent, since $x^2 - 2 = 1(x^2) - 2(1)$.

(c) Consider the solutions to $a(x+2) + b(x+1) + c(x^2 - 1) = \mathbf{0}$ (here $\mathbf{0}$ is the zero polynomial, $0x^2 + 0x + 0$). The resulting matrix equation can be written as $A\mathbf{x} = \mathbf{0}$, where $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{x} = (a, b, c)^T$. Since A is nonsingular, only the trivial solutions to the given equation exist, and it follows that the given polynomials are linearly independent in P_3.

3.3.7 We are to show that the given sets of vectors are linearly independent in $C[0, 1]$.

(b) The vectors are $x^{3/2}$ and $x^{5/2}$. In this case, the Wronskian is $W[x^{3/2}, x^{5/2}](x) = x^3$, which is nonzero except at the origin. These functions are linearly independent.

(d) The vectors are e^x, e^{-x}, and e^{2x}. The Wronskian in this case is

$$\begin{aligned} W[e^x, e^{-x}, e^{2x}](x) &= \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} \\ &= e^x \begin{vmatrix} -e^{-x} & 2e^{2x} \\ e^{-x} & 4e^{2x} \end{vmatrix} - e^{-x} \begin{vmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{vmatrix} + e^{2x} \begin{vmatrix} e^x & -e^{-x} \\ e^x & e^{-x} \end{vmatrix} \\ &= -6e^{2x}, \end{aligned}$$

which is everywhere nonzero. These are linearly independent.

3.3.11 We are to prove that any finite collection of vectors that contains the zero vector is linearly dependent. Going back to the definition, we must produce a nontrivial linear combination of the given vectors that gives us the zero vector. This is easy. Suppose that the collection in question is $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_k$, with $\mathbf{v}_k = \mathbf{0}$. Then letting $c_i = 0$ for each $1 \le i \le k-1$, and letting $c_k \neq 0$, we have $\sum_{i=1}^{k} c_i \mathbf{v}_i = \mathbf{0}$, showing linear dependence. $\square$

3.3.14 We are to show that if A is an $m \times n$ matrix with linearly independent columns, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This follows immediately from the definition of $A\mathbf{x}$. Letting $\mathbf{a}_i$ be the ith column of A, and setting $A\mathbf{x} = \mathbf{0}$, we have

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}.$$

Since the columns of A are linearly independent, only the trivial solution exists. $\square$

MA/Ra, October 7, 2004