Partial Solution Set, Leon §3.4

3.4.3 Given the vectors $\mathbf{x}_1 = (2, 1)^T$, $\mathbf{x}_2 = (4, 3)^T$, and $\mathbf{x}_3 = (7, -3)^T$,

(a) Show that $\mathbf{x_1}$ and $\mathbf{x}_2$ form a basis for $\mathbf{R}^2$.

(b) Why must $\mathbf{x_1}$, $\mathbf{x_2}$, and $\mathbf{x_3}$ be linearly dependent?

(c) What is the dimension of Span$(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$?

Solution:

(a) This follows because they are (by inspection) linearly independent in $\mathbf{R}^2$. Since the dimension of $\mathbf{R}^2$ is 2, $\mathbf{x_1}$ and $\mathbf{x}_2$ form a basis for $\mathbf{R}^2$.

(b) Since $\mathbf{R}^2$ is 2-dimensional, any collection of more than two vectors from $\mathbf{R}^2$ must be linearly dependent.

(c) Since $\mathbf{x_1}$ and $\mathbf{x}_2$ form a basis for $\mathbf{R}^2$, the dimension of Span$(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ is 2, i.e., Span$(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}) = \mathbf{R^2}$.

3.4.4 Given the vectors, $\mathbf{x}_1 = (3, -2, 4)^T, \mathbf{x}_2 = (-3, 2, -4)^T$, and $\mathbf{x}_3 = (-6, 4, -8)^T$, what is the dimension of Span$(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$?

Solution: Both $\mathbf{x_2}$ and $\mathbf{x_3}$ are scalar multiples of $\mathbf{x_1}$, so the dimension of Span$(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ is 1.

3.4.5 Given the vectors, $\mathbf{x}_1 = (2, 1, 3)^T, \mathbf{x}_2 = (3, -1, 4)^T$, and $\mathbf{x}_3 = (2, 6, 4)^T$,

(a) Show that $\mathbf{x_1}, \mathbf{x_2}$, and $\mathbf{x_3}$ are linearly dependent.

(b) Show that $\mathbf{x_1}$ and $\mathbf{x_2}$ are linearly independent.

(c) What is the dimension of Span$(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$?

(d) Give a geometric description of Span$(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$.

Solution:

(a) Letting $A = \begin{bmatrix} \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} \end{bmatrix}$, we consider the solutions to $A\mathbf{x} = \mathbf{0}$. (Yes, this is one of those situations in which the matrix turns out to be square, so the determinant is a possibility. But it is not recommended. Use Gaussian elimination instead.) Nontrivial solutions exist, so these three vectors are linearly dependent.

(b) For $\mathbf{x_1}$ and $\mathbf{x_2}$ to be linearly dependent, each must be a multiple of the other. This is clearly not the case, so they are in fact linearly independent.

(c) By (1), the dimension of Span$(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ is at most 2; by (2), the dimension of Span$(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ is at least 2. We have a squeeze play, and the dimension is 2.

(d) The subspace spanned by these three vectors is a plane through the origin in $\mathbf{R}^3$.

3.4.7 Find a basis for the subspace S of $\mathbf{R}^4$ consisting of all vectors of the form $(a+b, a-b+2c, b, c)^T$, where a, b, and c are real. What is the dimension of S?

Solution: Let $\mathbf{x}$ be an element of S. Then there exist real numbers a, b, and c such that

$$\mathbf{x} = \begin{bmatrix} a+b \\ a-b+2c \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Let $B = \left\{(1,1,0,0)^T, (1,-1,1,0)^T, (0,2,0,1)^T\right\}$. Since B spans S, and since B is a linearly independent set, it follows that B is a basis for S.

3.4.8 Given $\mathbf{x_1} = \mathbf{(1,1,1)^T}$ and $\mathbf{x_2} = \mathbf{(3,-1,4)^T}$:

(a) Do $\mathbf{x_1}$ and $\mathbf{x_2}$ span $\mathbf{R}^3$? Explain.

(b) Let $\mathbf{x_3}$ be a third vector in $\mathbf{R}^3$, and set $X = \begin{bmatrix} \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} \end{bmatrix}$. What conditions would X have to satisfy in order for the $\{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}\}$ to form a basis for $\mathbf{R}^3$?

(c) Find a third vector that extends the set $\{\mathbf{x_1}, \mathbf{x_2}\}$ to a basis for $\mathbf{R}^3$.

Solution:

(a) No. We know that $\mathbf{R}^3$ is 3-dimensional, since the standard basis for $\mathbf{R}^3$ contains three elements. But Span$(\mathbf{x_1}, \mathbf{x_2})$ has dimension 2 (they are linearly independent), and so must be a proper subspace of $\mathbf{R}^3$.

(b) The columns of X must be linearly independent.

(c) A random selection from $\mathbf{R}^3$ is almost guaranteed to miss the plane spanned by $\mathbf{x_1}$ and $\mathbf{x_2}$. (Almost, of course, is not acceptable. We must *verify* linear independence after we make our selection.) So let's pick, say, $\mathbf{x_3} = \mathbf{(0,0,1)^T}$. Gaussian elimination reveals that $\{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}\}$ is a linearly independent set.

3.4.11 Let S be the subspace of P_3 consisting of all polynomials of the form $ax^2+bx+2a+3b$. Find a basis for S.

Solution: Let $p \in S$. Then there exist reals a and b such that

$$p(x) = ax^2 + bx + 2a + 3b = a(x^2+2) + b(x+3).$$

Since $\{x^2+2, x+3\}$ is a linearly independent set (a quadratic polynomial cannot be a scalar multiple of a linear polynomial, and vice versa), it follows that $\{x^2+2, x+3\}$ is a basis for S.

3.4.12 In Exercise 3 of Section 2, some of the sets formed subspaces of $\mathbf{R}^{2\times 2}$. In each of these cases, find a basis for the subspace and determine its dimension.

Solution:

(a) A basis for the subspace of 2×2 diagonal matrices is the set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. The dimension is 2.

(b) A basis for the subspace of 2×2 lower triangular matrices is the set

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

The dimension of this subspace is 3.

(d) The set of all 2×2 matrices B such that $b_{11} = 0$. The dimension of this subspace is 3. A handy basis is the set $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

(e) The set of all 2×2 symmetric matrices. The dimension of this subspace is 3. For a basis, we might take $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.