Partial Solution Set, Leon §3.5

3.5.2c Let $\mathbf{u}_1 = (0,1)^T$ and $\mathbf{u}_2 = (1,0)^T$. Then $[\mathbf{u}_1, \mathbf{u}_3]$ is an ordered basis for $\mathbf{R}^2$. Find the transition matrix corresponding to the change of basis from the standard basis to $[\mathbf{u}_1, \mathbf{u}_3]$.

Solution: Let $U = [\mathbf{u}_1\mathbf{u}_2]$. Then U is the transition matrix corresponding to the change of basis from $[\mathbf{u}_1, \mathbf{u}_3]$ to the standard basis. It follows that U^1 is the matrix that we're after. Performing the computation, we find that $U^1 = U$, i.e., U is its own inverse.

3.5.4 Let $E = [(5,3)^T, (3,2)^T]$, and let $\mathbf{x} = (1,1)^T, \mathbf{y} = (1,-1)^T$, and $\mathbf{z} = (10,7)^T$. Find the values of $[\mathbf{x}]_E$, $[\mathbf{y}]_E$, and $[\mathbf{z}]_E$.

Solution: From our discussion in class, we know that the transition matrix from the ordered basis E to the standard basis is just

$$T = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}.$$

Since we want to go the other way, the transition matrix we want is

$$W = T^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}.$$

It is now an easy step to determine $[\mathbf{x}]_E = (-1,2)^T$, $[\mathbf{y}]_E = (5,-8)^T$, and $[\mathbf{z}]_E = (-1,5)^T$.

3.5.5 Let $\mathbf{u}_1 = (1,1,1)^T, \mathbf{u}_2 = (1,2,2)^T$, and $\mathbf{u}_3 = (2,3,4)^T$. Let $\mathbf{e}_i$ denote the ith standard basis vector. In part (a) of this problem, we are to find the transition matrix corresponding to the change of basis from $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ to $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$. We know that the transition matrix corresponding to the change of basis from $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ to $[\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_3]$ is $U = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. It follows that the matrix in which we are interested in is $U^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

In part (b), we are to compute the coordinates of various vectors with respect to $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$. This involves nothing more than matrix multiplication, and the solution is not included here.

3.5.6 We are given vectors $\mathbf{v}_1 = (4,6,7)^T, \mathbf{v}_2 = (0,1,1)^T$, and $\mathbf{v}_3 = (0,1,2)^T$, along with the set $\{\mathbf{u}_i\}$ of vectors from the preceding exercise. The task is to find the transition matrix from $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ to $[\mathbf{u}_1, \mathbf{u}_2\mathbf{u}_3]$. This is just $U^{-1}V$, where U^{-1} is the matrix found in the preceding problem and V is the matrix whose ith column is $\mathbf{v}_i$. We are then given the vector $\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3$, and asked to find its coordinates with respect to $[\mathbf{u}_1, \mathbf{u}_2\mathbf{u}_3]$. This is, once again, a matter of computing a product, in this case $U^{-1}V[\mathbf{x}]_V$, where $[\mathbf{x}]_V = (2,3,-4)^T$.

3.5.7 Given $\mathbf{v}_1 = (1,2)^T$, $\mathbf{v}_2 = (2,3)^T$, and $S = \begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix}$, find vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ such that S is the transition matrix from $\begin{bmatrix} \mathbf{w}_1, \mathbf{w}_2 \end{bmatrix}$ to $\begin{bmatrix} \mathbf{v}_1, \mathbf{v}_2 \end{bmatrix}$.

Solution: Let $W = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{bmatrix}$, and $V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$. It follows that $S = V^{-1}W$. But then

$$W = VS = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 9 & 4 \end{bmatrix}.$$

3.5.8 This is closely related to the previous problem. We are given $\mathbf{v}_1 = (2,6)^T$, $\mathbf{v}_2 = (1,4)^T$, and $S = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$. This time we want to find $\mathbf{u}_1$ and $\mathbf{u}_2$ such that S is the transition matrix from $\begin{bmatrix} \mathbf{v}_1, \mathbf{v}_2 \end{bmatrix}$ to $\begin{bmatrix} \mathbf{u}_1, \mathbf{u}_2 \end{bmatrix}$. We know that $S = U^{-1}V$. Multiplying from the left by U and from the right by S^{-1}, we have $U = VS^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}$. So $\mathbf{u}_1 = (0,-1)^T$ and $\mathbf{u}_2 = (1,5)^T$.

3.5.9 Let $[x, 1]$ and $[2x-1, 2x+1]$ be ordered bases for P_2. We want to find two transition matrices, one for changing basis from $[2x-1, 2x+1]$ to $[x, 1]$ and the other for changing basis from $[x, 1]$ to $[2x-1, 2x+1]$.

Solution: It is up to us to choose what the 'standard' basis is for P_2. We can choose this basis to be $[x, 1]$, with associated matrix $U = I_2$. Relative to this standard, the matrix associated with the other basis is $V = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$. Now the transition matrix for changing basis from $[2x-1, 2x+1]$ to $[x, 1]$ is V, while that for the inverse change is $V^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$.

3.5.10 Find the transition matrix representing the change of coordinates on P_3 from the ordered basis $[1, x, x^2]$ to the ordered basis $[1, 1+x, 1+x+x^2]$.

Solution: Let U denote the matrix for change of basis from the ordered basis $[1, 1+x, 1+x+x^2]$ to the (standard) ordered basis $[1, x, x^2]$. Finding U is relatively easy; the columns of U are simply the coordinate vectors (with respect to the standard basis) of the vectors in U. So we have

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix we want is $S = U^{-1}$. We can find S by performing elimination on the system

$\left[\begin{array}{c|c} U & I \end{array}\right]$, obtaining $\left[\begin{array}{c|c} I & S \end{array}\right]$. We obtain

$$S = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easily verified that this is the correct result.