Solutions to Selected Problems from West, Installment Two

2.1.7 Prove that every n-vertex graph with m edges contains at least $m - n + 1$ cycles.

Proof: We prove the result for connected graphs. The proof is by induction on m. If $m \leq n - 1$, then G is a tree and so contains $(n - 1) - n + 1 = 0$ cycles. Assume that $m = k > n - 1$, and that the result holds for graphs with fewer than $k - 1$ edges. Since $m > n - 1$, we know that G contains at least one cycle; let e be a cycle edge. By the induction hypothesis, $G - e$ contains at least $(m - 1) - (n - 1) = m - n$ cycles. Restoring e to G induces at least one additional cycle, so G has at least $m - n + 1$ cycles. $\square$

2.1.18 Prove that every tree with maximum degree $\Delta > 1$ has at least Δ vertices of degree 1.

Proof: Let T be a tree with maximum degree $\Delta \geq 2$. Let v be a vertex of degree Δ, and let $e_1, e_2, \ldots, e_\Delta$ be the edges incident to v. Let P_i $(1 \leq i \leq \Delta)$ be a maximal simple path leaving v on e_i. Since T is a tree, (i) P_i ends at a leaf v_i, and (ii) the leaves v_i $(1 \leq i \leq \Delta)$ are distinct. The result follows. $\square$

2.1.26 For $n \geq 3$, let G be an n-vertex graph such that every graph obtained by deleting one vertex is a tree. Determine $e(G)$, and use this to determine G itself.

Solution: Since every graph obtained by deletion of a single vertex is a tree, we know that G contains no cut vertex, so every vertex in G lies on a cycle, and that G contains n edges. It follows that $G = C_n$. $\square$

2.1.30 Let T be a tree in which all vertices adjacent to leaves have degree at least 3. Prove that T has some pair of leaves with a common neighbor.

Proof: Let P be a longest path in T, and let x be an endpoint of P. Necessarily, x is a leaf. Let w be the neighbor of x, and $y \in N(w) - V(P)$. We know that y exists because w has degree at least 3. Moreover, by our choice of P, y is a leaf. But then x and y are leaves with a common neighbor.

2.1.33 Let G be a connected n-vertex graph. Prove that G has exactly one cycle if and only if G has exactly n edges.

Proof: First suppose that G has exactly one cycle C, and let $e \in E(C)$. Then $G - e$ is a tree, and so has exactly $n - 1$ edges, and it follows that G has exactly n edges. Now suppose that G is connected and has exactly n edges. Since G is connected, G contains a spanning tree T which accounts for $n - 1$ edges of G. The remaining edge induces exactly one cycle. $\square$

2.1.72 Prove that if $T_1, T_2, \ldots, T_k$ are pairwise-intersecting subtrees of a tree T, then $\bigcup_{i=1}^{k} T_i \neq \emptyset$.

Proof: Let T be a tree. If $k = 2$, the result is immediate. Assume that $k > 2$, and that the result holds for all smaller sets of pairwise-intersecting subtrees of T. By hypothesis, there are vertices $x \in \bigcap_{i=1}^{k-1} T_i$, $y \in \bigcap_{i=2}^{k} T_i$, and $z \in T_1 \bigcap T_k$. Let P_{xy}, P_{xz}, and P_{yz} be the simple paths in T joining the indicated pairs of vertices. Note that $P_{xy} \bigcap P_{xz} \bigcap P_{yz} \neq \emptyset$, since otherwise we obtain a cycle in T. Moreover, since each subtree T_j $(1 \leq j \leq k)$ contains two of x, y, and z, it follows that each subtree contains at least one of P_{xy}, P_{xz}, and P_{yz}. But then

$$P_{xy} \bigcap P_{xz} \bigcap P_{yz} \subseteq \bigcap_{i=1}^{k} T_i,$$

and we're done. $\square$