Solutions to Selected Problems from West, Installment #3

3.1.8 Prove or disprove: Every tree has at most one perfect matching.

Proof: Let T be a tree, and suppose that T has perfect matchings M and M'. Let P be a maximal path in T. Then P must have odd length, since otherwise T has a leaf that is unsaturated. But then M and M' must agree on P. Since every edge in T lies on such a path, $M = M'$.

3.1.13 Let M and M' be matchings in an X,Y-bigraph G. Suppose that M saturates $S \subseteq X$ and M' saturates $T \subseteq Y$. Prove that G has a matching that saturates $S \bigcup T$.

Proof: We construct a matching M'' that saturates $S \bigcup T$, using selected edges of $M \bigcup M'$. Initially set $M'' := \emptyset$. Consider the subgraph H induced by $M \bigcup M'$. A component in H is either an isolated edge, an even cycle in which edges alternate between M and M' (an "alternating cycle"), or a path in which edges alternate between M and M' (an "alternating path"). For each isolated edge in $e \in M \bigcup M'$, set $M'' := M'' + e$. For each alternating cycle C in H, set $M'' := M'' + (E(C) \bigcap M)$. For each alternating path P of odd length, set $M'' := M'' + (E(P) \bigcap M)$ if the endpoints of P are M-saturated; otherwise set $M'' := M'' + (E(P) \bigcap M')$. For each alternating path P of even length, set $M'' := M'' + (E(P) \bigcap M)$ if both ends of P lie in X, and set $M'' := M'' + (E(P) \bigcap M')$ if both ends of P lie in Y. Since we did not take both M- and M'- edges from any component of H, M'' is a matching in G. It remains to show that $S \bigcup T$ is saturated. Let $v \in S \bigcup T$. If $v \in S \bigcup T$ is saturated by an isolated edge $e \in E(H)$, then since $e \in M''$ we know that v is M''-saturated. If v lies on an alternating cycle C in H, then v is saturated by one edge from each matching; the choice of $E(C) \bigcap M$ is arbitrary, and v is saturated by M''. If v is an internal vertex on an alternating path P of odd length, then v is saturated by both M and M', so we must consider only the case in which v is an endpoint of P. Since P has odd length, v and the opposite endpoint of P are either both $(M - M')$-saturated or both $(M' - M)$-saturated, and we choose accordingly to ensure that the endpoints of P are M''-saturated. Finally, suppose that v lies on an alternating path P of even length. As in the odd case, if v is an internal vertex on P then v is clearly M''-saturated, so suppose that v is an endpoint of P. If $v \in S$, then P is of the form $v = x_1, y_1, x_2, \ldots, y_{k-1}, x_k$ for some $k \geq 2$. It follows that $x_1 y_1 \in M$, and that $x_k \in X - S$; since the M-edges of P saturate every vertex of P except x_k, these are the edges that we place in M''. Similarly, if $v \in T$, then P is of the form $v = y_1, x_1, y_2, \ldots, x_{k-1}, y_k$ for some $k \geq 2$, $y_1 x_1 \in M'$, and $y_k \in Y - T$. Taking the M' edges of P saturates every vertex of P except y_k, and we're done. □

3.1.24 Prove that an $n \times n$ nonnegative integer matrix Q with constant row/column sum k can be expressed as a sum of k permutation matrices.

Proof: If $k = 1$, we're done, so assume that $k > 1$ and that the result holds for $k - 1$. Construct $G = (R, C, E)$, where $r_i c_j \in E$ iff $q_{ij} \neq 0$. Let M be a maximum matching in G. Set $p_{ij} = \begin{cases} 1; r_i c_j \in M \\ 0; \text{otherwise.} \end{cases}$ It is not hard to see that $P = (p_{ij})$ is a permutation matrix if and only if M is a perfect matching. It suffices to show that such a matching exists. We do so by showing that G satisfies Hall's condition. So let $S \subseteq R$. Suppose that $|N(S)| < |S|$. Then (relabeling the vertices if necessary) there exist $t < s$ such that $S = \{r_1, \ldots, r_s\}$ and $N(S) = \{c_1, \ldots, c_t\}$. Since the only nonzeros in rows $1, 2, \ldots, s$ of Q are in columns $1, 2, \ldots, t$, and since Q has constant line sum k, it follows that $\sum_{i=1}^{s} \sum_{j=1}^{t} q_{ij} = \sum_{i=1}^{s} k = sk$. But then the average of column sums $1, 2, \ldots, t$ is at least $sk/t > k$, a contradiction. So $|N(S)| \geq |S|$ for each $S \subset R$. A similar argument applies to subsets $S \subseteq C$. So M is a perfect matching, and P a permutation matrix. By the induction hypothesis, $Q - P$ is a sum of $k - 1$ permutation matrices, and the result follows. □

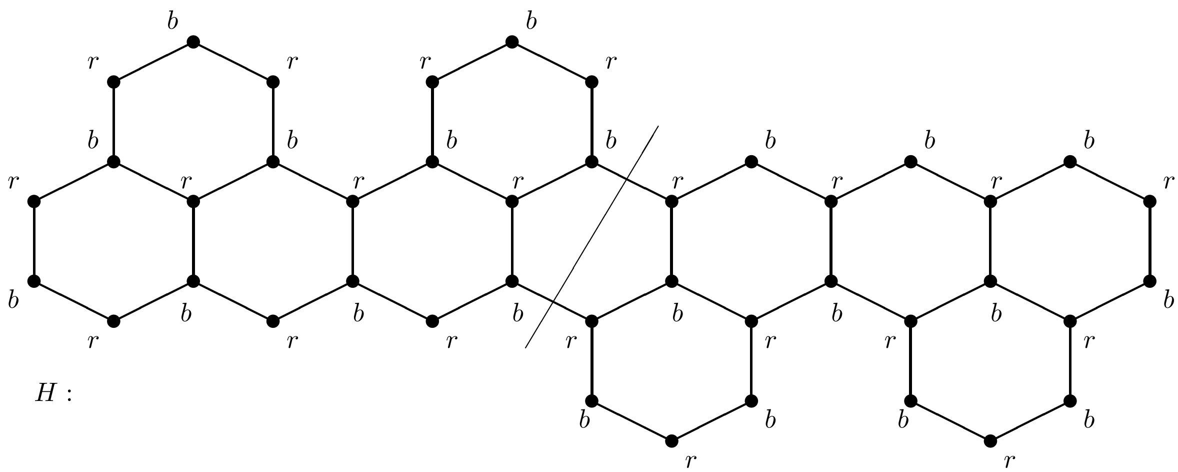

Figure 1: Figure for problem 3.1.28

3.1.28 We are to find either a perfect matching in the graph H, below, or a simple proof that none exists. The reality is that none exists. The graph is bipartite, and can be labeled as shown using colors red and blue. There are 21 vertices of each color, so at first glance it appears that a perfect matching might exist. Such a matching would necessarily contain 21 edges. But consider the edge cut shown by the line running southwest to northeast and bisecting the graph. The ten blue vertices to the left of the cut and the ten red vertices to the right constitute a vertex covering of cardinality twenty, so by the König-Egerváry theorem no perfect matching can exist. □

3.2.2 Show how to use the Hungarian Algorithm to determine whether a bipartite graph $G = (X, Y, E)$ has a perfect matching.

Solution: We can assume that $|X| = |Y|$, since otherwise no such matching exists. For the matrix of weights, we simply use the adjacency matrix of G. The Hungarian Algorithm will find a matching of total weight $|X|$ if and only if G has a perfect matching.

3.2.8 Suppose the weights in the $n \times n$ matrix A have the form $w_{ij} = a_i b_j$, where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are the weights associated with the rows and columns, respectively. Determine the maximum weight of a transversal of A. What about the case $w_{ij} = a_i + b_j$?

Solution: We first consider the problem in which $w_{ij} = a_i b_j$. We show that a greedy approach in which we iteratively attempt to maximize the product $a_i b_j$ among all available indices i, j is optimal. In other words, if $a_1 \leq a_2 \leq \cdots, \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$, then the diagonal traversal, with weight $\displaystyle\sum_{i=1}^{n} a_i b_i$, is optimal.

Without loss of generality (permuting rows and/or columns if necessary) assume that the inequalities described above hold. Let T be any transversal of A, and let D denote the diagonal transversal. We must show that $w(D) - w(T) \geq 0$. If $T = D$, there is nothing to show, so assume $T \neq D$. Then there exist $i_1, i_2, \ldots, i_k$ such that $a_{i_1 i_2}, a_{i_2 i_3}, \ldots, a_{i_k i_1} \in T$. We may assume that $i_1 = \min_k \{i_k\}$. It follows that

$$
\begin{aligned}
w(D) - w(T) &= w_{i_1 i_1} + w_{i_2 i_2} + \cdots + w_{i_k i_k} - (w_{i_1 i_2} + w_{i_2 i_3} + \cdots + w_{i_k i_1}) \\
&= a_{i_1} b_{i_1} + a_{i_2} b_{i_2} + \cdots a_{i_k} b_{i_k} - (a_{i_1} b_{i_2} a_{i_2} b_{i_3} + \cdots + a_{i_k} b_{i_1}) \\
&= a_{i_1}(b_{i_1} - b_{i_2}) + a_{i_2}(b_{i_2} - b_{i_3}) + \cdots + a_{i_k}(b_{i_k} - b_{i_1}) \\
&\geq a_{i_1}(b_{i_1} - b_{i_2}) + a_{i_1}(b_{i_2} - b_{i_3}) + \cdots + a_{i_1}(b_{i_k} - b_{i_1}) \\
&= a_{i_1} \sum_{j=1}^{k} (b_{i_j} - b_{i_j}) \\
&= 0,
\end{aligned}
$$

which is what we needed to show. □

Now consider the second problem, in which $w_{ij} = a_i + b_j$. Let T be any transversal of A. Then there exists a permutation π of $\{a_i\}$ such that T contains $a_{i\pi_i}$ for each $1 \leq i \leq n$. It follows that

$$
w(T) = \sum_{i=1}^{n} w_{i\pi_i} = \sum_{i=1}^{n} (a_i + b_{\pi_i}) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i,
$$

so all transversals have the same weight.