# Chapter Six Exercises

6.1.8 Prove that every planar graph has a vertex of degree at most five.

**Proof**: By Euler's identity, we know that every planar graph satisfies $e \leq 3n - 6$. If $\delta(G) \geq 6$, then $2e = \sum_v d(v) \geq 6n$, but then $e \geq 3n$. It follows that $\delta(G) \leq 5$. □

6.1.20 Prove, by induction on the number of faces, that a plane graph $G$ is bipartite if and only if every face of $G$ has even length.

**Proof**: First suppose that $G$ has a face $F$ of odd length. Then the boundary of $F$ includes a cycle of odd length, and $G$ is not bipartite. Now suppose that $G$ is a plane graph in which every face has even length. Since trees are bipartite, we can assume that $G$ has at least two faces. Let $xy$ be an edge shared by two faces, say $F_1$ and $F_2$, with respective lengths $l_1$ and $l_2$, and assume that the result holds for all plane graphs with both no faces of odd length and fewer faces than $G$. Notice that the induction hypothesis applies to $H = G - xy$, since $F_1$ and $F_2$ merge to form a single face with even length $l_1 + l_2 - 2$. It follows that we can 2-color the vertices of $H$. Moreover, since $d_H(x, y)$ is odd, $x$ and $y$ receive different colors in such a coloring. It follows that the coloring of $H$ is a proper coloring of $G$, and the result follows. □

6.1.25 Prove that every plane graph that is isomorphic to its dual has $2n - 2$ edges. For each $n \geq 4$, construct a self-dual plane graph.

**Proof**: We employ Euler's identity once again, this time in the form $n - e + f = 2$. If $G$ is a self-dual plane graph, then $n = f$, and the formula becomes $n = 2n - 2$. □

The construction: with $n = 4$ the only graph satisfying $n = 2n - 2$ is $K_4$, and this is the only complete self-dual graph. Moreover, a planar embedding of $K_4$ can be suggestively drawn as the triangular "wheel" $K_1 \vee C_3$. A bit of experimentation verifies that, for all $n \geq 4$, the wheel graph $K_1 \vee C_{n-1}$ is a self-dual plane graph with $n$ vertices.