EXAM I SOLUTIONS, MA4027, Summer 2004

Needless to say, there are other solutions.

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1. A *strongly regular graph*, or *SR*-graph, is a graph G that is regular of degree k, with v vertices, satisfying the following conditions:

   If p, q are adjacent vertices, then exactly λ vertices are adjacent to both p and q.

   If p, q are nonadjacent vertices, then exactly μ vertices are adjacent to both p and q.

   Such a graph G is said to have parameters (v, k, λ, μ). If both G and its complement are connected, G is said to be a nontrivial *SR*-graph, otherwise G is a trivial *SR*-graph. Show that the Petersen graph is nontrivial *SR*, and find its parameters.

   Solution: The Petersen Graph has ten vertices and is regular of degree three, so $v = 10$ and $k = 3$. The neighborhoods of adjacent vertices are disjoint, so $\lambda = 0$, while each nonadjacent pair has a common neighbor, so $\mu = 1$. Letting $G = (V, E)$ denote the Petersen graph, we now show that $\overline{G}$ is connected. Let $x, y \in V$, and assume that $xy \notin \overline{E}$. Since $|N(x)| = |N(y)| = 6$ and $|N(x) \bigcup N(y)| \leq 8$, it follows that $N(x) \bigcap N(y) \neq \emptyset$. Thus the Petersen graph is nontrivial SR with parameters $v = 10$, $k = 3$, $\lambda = 0$, and $\mu = 1$.

2. A saturated hydrocarbon is a molecule C_mH_n in which every carbon atom has four bonds, every hydrogen atom has one bond, and no sequence of bonds is cyclic. Give a graph-theoretic argument that, for every integer m, C_mH_n can exist only if $n = 2m + 2$.

   Proof: We can model a hypothetical saturated hydrocarbon with a graph G on $m + n$ vertices. Since carbon atoms have four bonds, we have m vertices of degree four, while the remaining n vertices represent hydrogen atoms and have degree one. The graph is a tree, since no sequence of bonds is cyclic. Since G is a tree, the number of edges in $E(G)$ is $m + n - 1$. By the degree-sum formula, the number of edges in $E(G)$ is $\frac{1}{2}(4m + n) = 2m + n/2$. Thus $m + n - 1 = 2m + n/2$, and it follows that $n = 2m + 2$. $\square$

3. Recall that a tournament is an orientation of K_n. A tournament G is *transitive* if, for all $x, y, z \in V(G)$, $xy \in E(G)$ and $yz \in E(G)$ imply $xz \in E(G)$; in other words, the adjacency relation is transitive. Prove that G is a transitive tournament if and only if G contains no directed cycle.

   Proof: Let G be a tournament. By definition, G is transitive if and only if G contains no cyclic triple. First suppose that G contains a cycle of length $k > 3$. If $k = 3$, we're done, so assume $k > 3$. Suppose C is a directed cycle on vertices $v_1, v_2, \ldots v_k$. Let j be the first index in $3, 4, \ldots, k$ such that $v_jv_1 \in E(G)$. (Note that j is well-defined, since

$v_k v_1 \in E(G)$.) Then $v_1 v_{j-1} v_j v_1$ is a directed 3-cycle. It follows that if G contains a directed cycle of any length, then G contains a cyclic triple and is not transitive. Now suppose that G contains no cycles of any length. Then clearly G contains no cyclic triple, and thus G is transitive. □

4. Prove that G is bipartite if and only if every subgraph H of G contains an independent set consisting of at least half of $V(H)$.

   Proof: First suppose that $G = (X, Y, E)$ is bipartite. Let $H = (X', Y', E')$ be a subgraph of G. Then H is bipartite. The larger of X', Y' (choose arbitrarily if $|X'| = |Y'|$) is an independent set containing at least half of $V(H)$. Now suppose that G is not bipartite. Then G contains a subgraph $H \cong C_{2k+1}$. Since any set of more than k vertices of H must contain two adjacent vertices, no independent set in H contains half of $V(H)$. □

5. Use the König-Egerváry Theorem (Theorem 3.1.16) to prove Hall's Theorem (Theorem 3.1.11).

   Solution: The theorem to be proven states that, for any bipartite graph $G = (X, Y, E)$, G has an X-saturating matching iff $|N(S)| \geq |S|$ for every $S \subseteq X$. The König-Egerváry Theorem states that if $G = (X, Y, E)$ is bipartite, then the size of a smallest vertex cover is equal to that of a largest matching.

   We now proceed with the

   Proof: Let $G = (X, Y, E)$. If $|N(S)| < |S|$ for some $S \subseteq X$, then no S-saturating matching, and therefore no X-saturating matching, can exist. We must prove the converse. Suppose, then, that $|N(S)| \geq |S|$ for every $S \subseteq X$. It suffices to show that $|K| \geq |X|$ for every cover K, since then by the König-Egerváry Theorem $|M| \geq |X|$ for every maximum-cardinality matching M. So let K be a cover in G, and let $S = X - K$. Since K is a cover, and since $S \nsubseteq K$, then we know that $N(S) \subseteq K$. It follows that

$$|K| \geq |X - S| + |N(S)| = |X| - |S| + |N(S)| \geq |X|,$$

   and the proof is complete. □