<u>Halley's method</u> (1694) (the Halley, of comet fame).

Idea: approximate $f(x)$ locally by an LFT (<u>linear fractional transformation</u>):

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \tfrac{1}{2} f''(x_0)(x-x_0)^2 + \cdots$$
$$\doteq \frac{a + b(x-x_0)}{c + d(x-x_0)} \quad \text{near } x = x_0 .$$

Set

$$z := x - x_0, \quad x := x_0 + z .$$

Want

$$\frac{a+bz}{c+dz} = f_0 + f_0' z + \tfrac{1}{2} f_0'' z^2 + \cdots$$

with obvious notation, that is

$$a + bz = (c + dz)\left(f_0 + f_0' z + \tfrac{1}{2} f_0'' z^2 + \cdots\right)$$
$$= c f_0 + c f_0' z + \tfrac{1}{2} c f_0'' z^2 + \cdots$$
$$+ d f_0 z + d f_0' z^2 + \cdots ,$$

so

$$a = c f_0, \quad b = c f_0' + d f_0, \quad 0 = \tfrac{1}{2} c f_0'' + d f_0' .$$

Next iterate to a zero of f is $x_1 = x_0 + \Delta x_0$ with $a + b\Delta x_0 = 0$, so $x_1 = x_0 - \frac{a}{b}$. Now

$$\frac{a}{b} = \frac{c f_0}{c f_0' + d f_0} = \frac{f_0}{f_0' + \frac{d}{c} f_0}$$

so Halley's algorithm is a "correction"

of Newton's. We need only d/c which follows from the coefficient of z^2 above. Since $cf_0'' + 2d f_0' = 0$ then $d/c = -\frac{1}{2} f_0''/f_0'$, so the iteration is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k) - \frac{1}{2}\frac{f(x_k)f''(x_k)}{f'(x_k)}}.$$

<u>Problem</u>. Show that Halley's iteration for $f(x) = x^n - c$, $c > 0$, is <u>Lambert's algorithm</u> <u>of</u> 1770 for computing $c^{1/n}$:

$$x_{k+1} = x_k \frac{(n+1)c + (n-1)x_k^n}{(n-1)c + (n+1)x_k^n}.$$

<u>Problem</u>. Write Halley's algorithm in the form

$$x_{k+1} = g(x_k)$$

with <u>iteration</u> <u>function</u> g. Show that $f(x^*) = 0 \Leftrightarrow x^* = g(x^*)$, so x^* is a <u>fixed point</u> of g. Also show that $0 = g'(x^*) = g''(x^*) \neq g'''(x^*)$, in general. Hence, with the errors $e_k = x_k - x^*$,

$$e_{k+1} = \frac{1}{6} g'''(c_k) e_k^3, \quad c_k \text{ between } x^* \text{ and } x_k,$$

so the method is of order ≥ 3.

Halley's method for inverse functions.

$f(g(x)) \equiv x$, x in domain of g

To compute $y = g(x)$ solve

$$F(y) := f(y) - x = 0$$

for $y = g(x)$ using Halley's algorithm.

$$y_{k+1} = y_k - \frac{F(y_k)}{F'(y_k) - \frac{1}{2}\frac{F(y_k)F''(y_k)}{F'(y_k)}}$$

Example: $f(x) = \ln x$, $y = g(x) = e^x$.

$$y_{k+1} = y_k \frac{2 + x - \ln y_k}{2 - x + \ln y_k},$$

$$k = 0, 1, 2, \ldots.$$

Problem. What about its global convergence?

Example. $x = y_0 = 1$

$y_0 = 1$,

$y_1 = 3$,

$y_2 = 2.718064296486053$,

$y_3 = 2.718281828459161$,

$y_4 = 2.718281828459046$,

$y_5 = 2.718281828459046$.